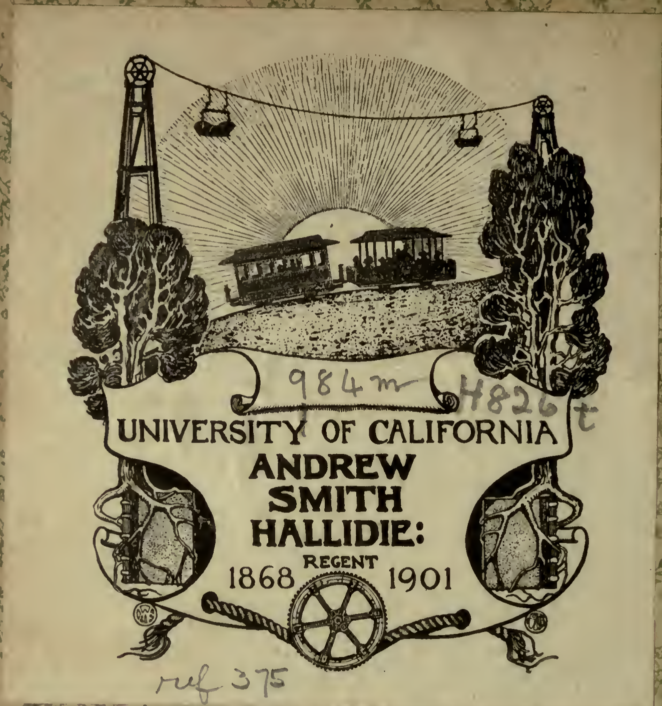

UNIVERSITY OF CALIFORNIA
ANDREW SMITH HALLIDIE:
REGENT
1868 1901

THEORETICAL MECHANICS

STANFORD UNIVERSITY BOOKSTORE
Agent of the Publisher

Theoretical Mechanics

An Elementary Text-book

BY

L. M. Hoskins

Professor of Applied Mathematics in the
Leland Stanford Junior University

SECOND EDITION

STANFORD UNIVERSITY, CAL.
PUBLISHED BY THE AUTHOR
1903

Stanford University Press

PREFACE.

In a first course in Theoretical Mechanics the primary object to be gained by the student is a thorough grasp of fundamental principles. In most cases it is impossible to go beyond this object in the time available for the course. In the preparation of this text-book, the aim has been to present the fundamental principles in as clear and simple a manner as possible, and to enforce them by a sufficient number of illustrative examples. The examples are for the most part simple applications of the theory presented in the text, many numerical exercises being included. The solution of exercises involving numerical data forms a part of the work of the student of which the importance should be emphasized. In the desire to cover as much ground as possible, such work is too apt to be neglected.

The mathematical training required for using the book is that usually implied by an elementary knowledge of Differential and Integral Calculus. The most essential portions of Part I (Statics) may, however, be read without such knowledge.

As regards arrangement, the aim has been to observe the order of difficulty of the different topics treated, so far as this was possible without doing violence to the logical relations. This is one reason for placing Statics first, instead of treating it in its strictly logical position as a special case of Kinetics. Part II may, however, be read before Part I, if any teacher should prefer this order.

As to scope, the primary object has been to meet the needs of students of engineering in American universities and technical colleges. More has, however, been included than it will usually be possible or desirable to cover in an elementary course. The precise limits of the course must be determined by the judgment of the teacher, taking due account of the relative importance of the various parts of the subject and of the limitations of time. It is hoped that the arrangement of the book is such as to facilitate whatever abridgment the teacher may find desirable.

In case it is necessary to cut the course down to the narrowest possible limits, the following suggestions as to the most essential parts to retain may be of service.

Chapters I–V. These include, besides the introductory general matter, the fundamental principles governing the composition and resolution of forces and the conditions of equilibrium, as applied to a rigid body acted upon by coplanar forces.

Chapters XII–XVII; omitting some of the applications in Chapters XIII and XV, and § 2 of Chapter XVI. These chapters cover the most essential parts of the theory of the motion of a particle, including a general discussion of the laws of motion and an introduction to the theory of energy.

Chapters VI and VII are of practical value to the student of engineering, and the course should include at least § 1 and § 2 of Chapter IX.

Chapters XVIII–XX constitute an introduction to the Dynamics of Material Systems, with applications to the rotation of a rigid body. If possible, at least this portion of Part III should be included in the course. In Chapter XIX, § 2 (methods of computing moments of inertia of plane areas) is of practical value for its application in Mechanics of Materials, and may be taken independently of any other portion of Part III.

If this book is in any degree successful in meeting the needs of students of engineering, it is hoped that it may be of service also to those pursuing the subject for its intrinsic scientific interest or as a preparation for the study of mathematical physics. The opinion is sometimes expressed that the needs of these different classes of students require essentially different methods of treating the subject. This view, so far as it refers to the fundamental parts of an elementary course, is not shared by the author of this text-book. For all students, the matter of first importance is the clear understanding of fundamental general principles and the ability to apply them.

STANFORD UNIVERSITY, CAL., October, 1900.

PREFACE TO SECOND EDITION.

In this edition Chapters I–XXIII are reprinted from the first edition without important change, except the correction of such typographical and other errors as have been detected. Chapter XXIV is new.

STANFORD UNIVERSITY, CAL., August, 1903.

CONTENTS.

PART II. MOTION OF A PARTICLE.

PART III. MOTION OF SYSTEMS OF PARTICLES AND OF RIGID BODIES.

THEORETICAL MECHANICS.

CHAPTER I.

INTRODUCTORY.

§ 1. *Preliminary Notions.*

1. **Mechanics Defined.**— *Mechanics* is the science which treats of the motions of material bodies.

Motions take place in accordance with definite laws. This statement means that the motions of bodies depend in an invariable way upon certain definite conditions. The fundamental object of the science of Mechanics is to investigate these conditions and to formulate the laws in accordance with which they determine the motion.

In its present form the science of Mechanics rests upon a few fundamental laws of great generality. These laws are generalizations from experience, and in the presentation of the principles of the science they are taken as postulates,— propositions not deducible from anything more fundamental. So far as these postulates and the principles deduced from them give a true account of the motions of natural bodies, they constitute a *natural science.*

These fundamental laws, however, involve certain conceptions of matter and of motion which are ideal. This is necessarily the case. The motion of a body can be completely specified only by describing the motion of every ultimate portion of which it is composed; yet the ultimate structure of matter is wholly unknown. The bodies to which the laws of motion apply are therefore defined in an ideal way. Moreover, the very conception of motion involves the abstract notions of Geometry, with the notion of time added. Because of this ideal character of the laws, the science based upon them is properly called *Theoretical Mechanics.*

2. Material Bodies.— A *body* is any definite portion of matter.

No attempt need be made to define matter or to enumerate its properties. A sufficiently definite preliminary notion is supplied by ordinary experience.

The characteristics of material bodies which are of importance in a study of their motions are the following: (1) Every body has a definite *volume* and a definite *figure;* (2) every body possesses a definite *mass;* (3) bodies exert *forces* upon one another.

3. Mass.— The *mass* of a body is often briefly defined as its "quantity of matter." These words, however, convey no definite idea of the meaning of mass as a factor in the determination of motion. A satisfactory definition of mass cannot be given in advance of a discussion of the fundamental laws of motion.

We conceive of mass as the one invariable characteristic of matter. Every individual portion of matter is regarded as possessing a definite mass whose value is uninfluenced by changes of position or by physical or chemical transformations. The volume and shape of a body may change, the forces it exerts upon other bodies and those which they exert upon it are different under different circumstances; but its mass is regarded as an absolute constant.

4. Force.— A *force* is an action exerted by one body upon another, tending to change the state of motion of the body acted upon.

A force may be conceived as a push or a pull, acting upon a definite portion of a body. Such a push or pull always tends to change the motion of the body; but this tendency may be counteracted in whole or in part by the action of other forces.

Mechanics is often called the *science of motion and force*, because of the importance of force in the development of the laws of the science.

Force, like mass, is a quantity whose significance cannot be satisfactorily explained except by a full discussion of the fundamental laws of motion.

5. Particle.— A body may be conceived to be divided into very small parts, each of which may be called a particle. Ideally, there is no limit to this process of subdivision. For the purposes of mathematical analysis it is often conceived to be carried so far that the linear dimensions of the particles become vanishingly small.

These particles may be conceived in either of two ways, corresponding to two different conceptions of the structure of matter.

(1) It may be assumed that matter occupies space *continuously*. By this it is to be understood that every portion of matter whose mass is finite occupies a finite volume.

(2) It may be assumed that any definite portion of matter consists of particles, each of which possesses finite mass but occupies no finite volume.

The hypothesis of continuity does not necessarily imply that there may not be void spaces between the parts of a body. In mathematical language its meaning may be stated as follows: Let M be the mass of a body and V its volume; and let ΔM be the mass of a small portion whose volume is ΔV. Then if ΔV is taken smaller and smaller so that $\Delta V/V$ approaches zero, the hypothesis of continuity implies that $\Delta M/M$ also approaches zero; while the hypothesis of discontinuity implies that $\Delta M/M$ may be finite.

In certain discussions it is found convenient to adopt one hypothesis, in other cases the other; so far as results are concerned, it is usually immaterial which is adopted.

In the analysis of the motion of a particle, it is regarded as a geometrical point endowed with mass. A body whose linear dimensions are small in comparison with the range of its motion is often regarded as a particle.

6. Rigid Body.— A *rigid body* may be defined as a body whose particles do not change their distances from one another.

An important part of the science of Theoretical Mechanics deals with bodies which are assumed to satisfy this definition.

Actual solid bodies undergo appreciable changes of shape and size; and if sufficiently small portions could be observed, they would doubtless be found to be in rapid motion, thus departing very far from the condition specified in the definition of rigidity. But disregarding the motions of ultimate particles and considering only the motion of a body as a whole, the theory of the motion of an ideal rigid body describes with great accuracy the motion of an actual solid body whose shape remains nearly constant.

7. Position and Motion.— In the foregoing definitions and explanations it has been assumed that position and motion need no definitions. It is, in fact, doubtful whether any definitions can be given which convey clearer notions than the words themselves.

By the position of a particle is meant its relation in space to some body taken as a standard of reference.

A particle is in motion when its position is changing. A body is in motion when the particles composing it are in motion.

What shall be chosen as the body of reference in specifying the position and motion of a particle is a matter of arbitrary choice. Position and motion are thus not absolute but relative. The motion of a particle with reference to one body may be very different from its motion with reference to another. (See Arts. 267, 268.)

8. **Kinds of Quantity.**— The science of Mechanics deals with quantities of four fundamental kinds : time, space, mass, force. In the development of the principles of the science other quantities are introduced which are derived from these but involve no other elementary conceptions.*

9. **Divisions of the Subject.**— The general subject of Mechanics may be subdivided in various ways.

First, the basis of the subdivision may be the *nature of the bodies* dealt with. On this basis there are the following divisions :

(*a*) *Mechanics of a Particle* and of systems of particles in general.

(*b*) *Mechanics of Solid Bodies*, including (1) rigid and (2) non-rigid solids.

(*c*) *Mechanics of Fluids*, including (1) liquids and (2) gases.

The present work deals mainly with particles and with rigid solids.

Second, the basis of subdivision may be the *fundamental kinds of quantity* involved. On this basis the whole subject, and each of the above divisions, may be divided as follows :

(*a*) *Kinematics*, treating of motion, without reference to the causes producing or influencing it. Under this there are (1) Pure Kinematics, treating of motion apart from the idea of mass, and (2) Mass-Kinematics, treating of motion and mass.

(*b*) *Dynamics*, treating of forces and of their influence upon the motions of bodies. Dynamics, the science of force, is again subdivided into Statics and Kinetics.

---

* It is pointed out in a later chapter that these four quantities are not always regarded as being all fundamental. In particular, force is often regarded as a quantity which must be defined in terms of mass, space and time. Ordinary experience, however, supplies a tolerably definite notion of force which is wholly independent of any conception of the meaning of the laws of motion. See "The Six Gateways of Knowledge," by Lord Kelvin.

Statics treats of the conditions of equivalence of systems of forces, and especially of the conditions under which forces are balanced so that they do not affect the motions of the bodies acted upon. *Kinetics* treats of the laws in accordance with which the motions of bodies are influenced by the forces acting upon them and by their masses.

Strictly speaking, Kinetics includes Statics; for from a knowledge of the effects of forces upon the motions of bodies may be derived all principles relating to the equivalence of different systems of forces and to the conditions under which forces are balanced. Statics and Kinetics are, however, often treated as coördinate branches of Dynamics; first, because Statics, although a special case, is a case of great importance, and second, because the principles of Statics can be developed to a large extent independently of those of Kinetics.

The arrangement of subjects in this book does not strictly follow either of the above classifications, but is designed to meet the needs of beginners by presenting the different topics somewhat in the order of their difficulty. The arrangement adopted is as follows:

Introductory chapter, treating of certain principles which have application in various branches of the general subject of Mechanics.

Part I. Statics; the discussion of the subject being limited mainly to systems of forces acting in the same plane upon a particle or a rigid body.

Part II. Motion of a Particle; including Kinematics and Kinetics, but limited mainly to the case of motion in a plane.

Part III. Motion of Systems of Particles and of Rigid Bodies; including Kinematics and Kinetics, but dealing mainly with motion in a plane.

The remaining portion of this introductory chapter is devoted to certain principles which, while not strictly included in the subject of Mechanics, are of fundamental importance in the development of the science.

§ 2. *The Numerical Representation of Quantities.*

10. Comparison of Like Quantities.— The magnitudes of any two quantities of the same kind (as two distances, or two intervals of time, or two forces) may be compared by determining their *ratio.* Such a ratio is an abstract number.

11. Numerical Value of a Quantity.— The numerical value of

a quantity is the ratio of its magnitude to that of a given quantity of the same kind, taken as a standard. This standard quantity is called a *unit*. Thus, to express a length as a number, some definite length must be chosen as a unit (its numerical value being called *one*); then the numerical value of any given length will be the number expressing its ratio to the unit length.

The unit in terms of which quantities of a given kind are expressed is wholly arbitrary. Thus, there are in common use several different units of length, as the foot, the inch, the yard, the mile, the meter; several units of time, as the second, the minute, the week, the year; and several units of mass, as the pound, the ton, the ounce, the gram, the kilogram.

12. Numerical Value Depends on Unit.— Evidently, the numerical value of a quantity gives no idea of its magnitude unless the unit employed is known. Any given quantity may be represented by any number whatever, by properly choosing the unit. Thus, the same distance may be called 10,560 feet, 3,520 yards, or 2 miles.

But the *ratio* of two quantities of the same kind is a definite number, which depends only upon the magnitudes of the quantities, and not upon their numerical values. The ratio of the numerical values of two quantities is the same as the ratio of the quantities themselves *if they are expressed in the same unit*, but not otherwise.

13. Fundamental and Derived Units.— Although, as above stated, the unit in terms of which quantities of any particular kind are expressed numerically may be chosen arbitrarily, it is usually advantageous to choose the units for quantities of different kinds in such a way that certain of them depend upon others. In Physics, the usual practice is to choose arbitrarily the units of time, length and mass, and to make all other units depend upon these. Those units which are chosen arbitrarily are called *fundamental*, while those which are so defined as to depend upon the fundamental units are called *derived* units.

For example, the unit of area is usually taken as the area of a square whose side has the unit length. It should, however, be clearly understood that this dependence is not necessary, but is merely assumed for convenience.

14. Dimensions of Derived Units.— The word *dimension* is commonly employed to describe the way in which a derived unit depends upon the fundamental units.

Thus, if the unit area is defined as the area of a square whose side has the unit length, the relation may be expressed by the "dimensional equation"

$$(\text{unit area}) = (\text{unit length})^2,$$

and the unit area is said to involve two dimensions of length. Similarly, if the unit volume is defined as the volume of a cube whose edge has the unit length, the unit volume involves three dimensions of length.

In the illustrations just given, only one kind of fundamental quantity (length) is employed. There will be occasion in the following pages to deal with derived units which depend upon two or more fundamental units.

Evidently the manner in which a derived unit is made to depend upon the fundamental units is to some extent arbitrary, and by changing the manner of dependence the dimensions of any given kind of quantity may be changed. This subject will be further treated in connection with the various kinds of quantity dealt with in the following pages.

15. Dimensional Equations.— The dimensional equation above written is not to be interpreted as an ordinary algebraic equation. It is merely a concise method of expressing the way in which the derived unit (of area) is made to depend upon the fundamental unit (of length). It will be convenient to abbreviate such equations by the use of symbols to denote the various units. The quantities whose units are taken as fundamental in the following pages are usually some or all of the following: length, mass, time, force. In writing dimensional equations the letters **L, M, T, F** will be used to designate the units of these four kinds of quantity.

§ 3. *Vector Quantities.*

16. Vector Quantity Defined.— A directed straight line of definite length is called a *vector*.

A *vector quantity* is one possessing *direction* as well as magnitude.

Any vector quantity may be represented by a vector. For this purpose the direction of the vector must agree with that of the quan-

tity to be represented, and its length must represent the magnitude of the quantity.

In order that the length of a line may represent the magnitude of any given quantity, a *scale* must be chosen. That is, a certain length must be taken as equivalent to a unit magnitude of the kind to be represented. The length thus chosen is arbitrary, but when once chosen, the length representing any definite magnitude of the kind in question is determined.

A vector may be designated by naming, in proper order, the letters placed at its ends. Thus, if A and B are any two points, AB and BA denote two vectors of equal length but opposite in direction.

A vector may also be designated by a single symbol, as in the case of ordinary algebraic quantities.

Scalar.— The processes of ordinary algebra are limited to quantities which either do not involve direction or else are restricted to two opposite directions. Such quantities are called *scalars.* A scalar quantity is completely specified by assigning its magnitude and algebraic sign.

17. Equality of Vectors.— Two vectors are said to be equal if they agree not only in magnitude but in direction. Thus, if MN and PQ (Fig. 1) are parallel and of equal length, we may write

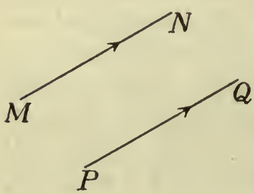


FIG. 1.

$$MN = PQ.$$

If this were an ordinary algebraic equation, it would express only equality of length, and would be equally true if the lines were not parallel; but as a vector equation it expresses identity of direction as well as equality of length. Thus, referring to Fig. 1,

$$MN = QP$$

is not true as a vector equation.

18. Addition of Vectors.— Non-parallel vectors cannot be added in the sense in which scalars are added; but the word addition is used to describe the process of combining two vectors in the following manner:

Let MN and PQ (Fig. 2) represent two vectors; then by their sum is meant a vector determined as follows: Make AB equal and parallel to MN, and BC equal and parallel to PQ; then AC represents the required sum. In fact, there may be written the equation

$$AB + BC = AC; \text{ or } MN + PQ = AC.$$

But this must be understood as a *vector equation*, and not as applying to the *lengths* MN, PQ and AC.

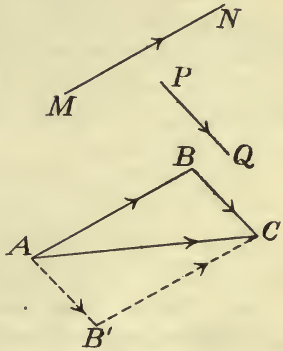


FIG. 2.

If the order of the two vectors is changed their sum remains the same. For if lines equal and parallel to PQ and MN be laid off successively from A, the result will be AB' and $B'C$, which, with AB and BC, form a parallelogram of which AC is a diagonal.

Next, let any three vectors be drawn consecutively, as AB, BC, CD (Fig. 3); then AD is called their sum. It is easily seen that the three vectors may be drawn in any order without changing their sum as thus defined. It is also evident that

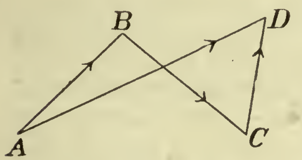


FIG. 3.

$$\begin{aligned} AB + BC + CD &= AC + CD \\ &= (AB + BC) + CD; \end{aligned}$$

that is, the result of adding the three vectors is the same as the result of adding any one to the sum of the other two.

The same construction may be extended to any number of vectors.

19. Vector Subtraction.— Parallel vectors may be distinguished in direction by signs plus and minus as in Algebra. Thus, if A and B are any two points, the two vectors AB and BA may be designated by prefixing signs plus and minus to the same symbol; if the symbol p represents the vector AB, $-p$ represents the vector BA.

The minus sign may thus be used to denote reversal of the direction of a vector, so that we may write

$$AB = -BA.$$

It may also be used to denote subtraction, in accordance with the following definition:

To subtract a vector is to add its negative.

With this definition of subtraction, the difference between two vectors may be found as follows:

Let it be required to subtract the vector MN from the vector PQ (Fig. 4). Lay off $AB = MN$ and $AC = PQ$; then

$$\begin{aligned} PQ - MN &= AC - AB \\ &= AC + BA = BA + AC \\ &= BC. \end{aligned}$$

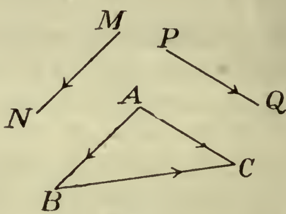


FIG. 4.

20. Vector Equations.— From the above definitions of vector addition and subtraction, vector equations may be formed containing any number of terms in each member, and may be transformed in accordance with simple rules. Thus, if A, B, C, D, E, F are any six points, we may write the vector equation

$$AB + BC + CD = AE + EF + FD; \quad (1)$$

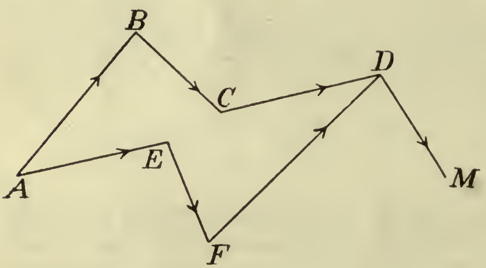


FIG. 5.

for each member of the equation is equal to the vector AD.

In several particulars vector equations may be treated by the same rules as the equations of ordinary Algebra.

(1) Equal vectors may be added to each member of a vector equation without destroying the equality. Thus, if a vector DM be added to each member of equation (1), each member becomes equal to AM, and therefore the equality still holds. By an extension of this principle, any number of vector equations may be added, member to member.

(2) A term may be transposed from one member to the other by changing its sign. Thus, from equation (1) we may derive the equation

$$AB + BC = AE + EF + FD + DC,$$

or $$AB + BC = AE + EF + FD - CD,$$

by transposing the term CD and changing its sign.

(3) The order of the terms in either member of a vector equation may be changed at pleasure. For, as already pointed out, changing the order in which several vectors are combined does not change their sum.

21. **Constant and Variable Vector Quantities.**— A vector quantity may be either *constant* or *variable.*

A *constant* vector quantity remains unchanged not only in magnitude but in direction. A *variable* vector quantity changes either in magnitude or in direction, or in both.

If a point A is fixed, while another point B moves in any manner, the vector AB is variable. If B moves always along the straight line which at a certain instant passes through the two points, the vector varies in magnitude only. If B describes a circle with center at A, the vector varies in direction but not in magnitude. If B describes any other path, the vector varies both in magnitude and in direction.

22. **Increment of Variable Vector.**— If a variable vector has different values at the beginning and end of any interval of time, its *increment* during that interval is the vector which, added to the initial value, will produce the final value. The increment may be determined, when the initial and final values of the vector are known, by the following construction:

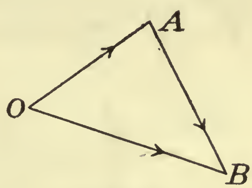


FIG. 6.

From any point O (Fig. 6), lay off OA and OB, representing the initial and final values of the variable vector; then AB represents the required increment. For, there may be written the vector equation

$$OA + AB = OB,$$

showing that AB satisfies the definition of increment just given.

The increment is evidently found by subtracting the initial value of the variable vector from the final value. For

$$OB - OA = AB.$$

In general, if p_1 denotes the initial value of a vector and p_2 its final value, the vector increment is equal to $p_2 - p_1$, the minus sign denoting *vector* subtraction (Art. 19).

23. Resolution of a Vector.—To *resolve* a vector is to express it as the sum of any number of vectors. If AB represents the given vector, it can be resolved by constructing a closed polygon of which AB is a side; thus,

$$AB = AL + LM + MN + NB,$$

L, M, and N being any three points whatever.

Any vectors whose sum is equal to a given vector are called *components* of that vector.

The discussion which follows will be limited mainly to *coplanar* vectors; that is, to vectors lying in, or parallel to, the same plane. The foregoing principles are, however, true without this restriction.

24. Resolution into Two Vectors Having Given Directions.—A given vector can be expressed as the sum of two vectors parallel to any two lines coplanar with it. For, let AB (Fig. 7) represent the given vector, and MN, PQ the given lines. From A draw a line parallel to MN, and from B a line parallel to PQ. Let C be the point of intersection of these lines; then $AC + CB = AB$, and AC and CB are the required components.

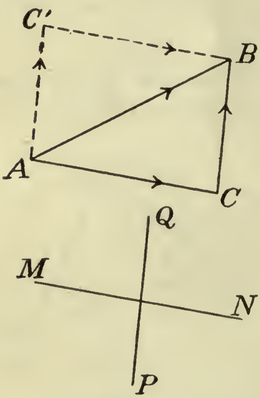


Fig. 7.

An equally correct construction is to draw AC' parallel to PQ, and $C'B$ parallel to MN. The result is the same as before, since AC' and CB are equal vectors, as are also AC and $C'B$.

Evidently only one pair of components can be found which are equivalent to a given vector and have assigned directions.

25. Resolved Part of a Vector.— If AC and CB (Fig. 8) are at right angles to each other, each is called the *resolved part* of AB.

Fig. 8.

If AB represents any vector quantity, and θ is the angle between AB and a given direc-

tion, the resolved part of AB in that direction is given in magnitude by the product

$$(\text{magnitude } AB) \times (\cos\theta).$$

The resolved part of a vector is equal to the algebraic sum of the resolved parts of its components, for any direction of resolution. Thus (Fig. 9) AE is the sum of the vectors AB, BC, CD, DE. The resolved parts of these components parallel to the line MN are $A'B'$, $B'C'$, $C'D'$, $D'E'$; while the resolved part of AE is $A'E'$. Evidently $A'E'$ is the algebraic sum of $A'B'$, $B'C'$, $C'D'$, $D'E'$.

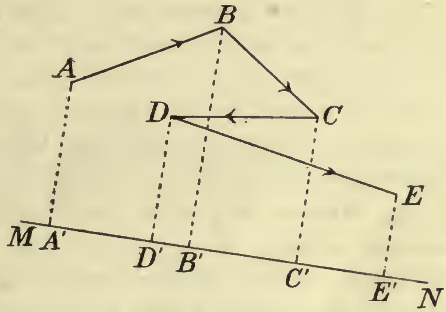


FIG. 9.

26. Localized Vector.— If a vector is associated with a definite point or line in space, it may be called a *localized vector.*

The point at which a vector is localized may be called its *position;* and the line in which it is localized its *position-line.*

27. Moment of Localized Vector.— The *moment* of a localized vector about a point is the product of the magnitude of the vector into the perpendicular distance of its position-line from the given point.

The *origin of moments* is the point about which the moment is taken. The *arm* of the vector is the perpendicular distance of its position-line from the origin.

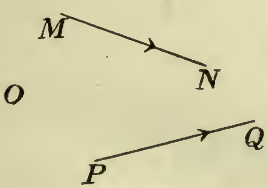


FIG. 10.

The *plane of the moment* is the plane containing the origin of moments and the position-line of the vector.

In comparing moments having the same plane (or parallel planes) there are two cases which may be distinguished by signs plus and minus. These cases are illustrated in Fig. 10, in which O is the origin of moments and MN, PQ denote the directions and position-lines of two localized vectors. The relations of MN and PQ to the origin O

suggest rotations about O in opposite directions. One of these directions being taken as plus, the other will be called minus. Which shall be called plus may be decided arbitrarily. For uniformity, it will usually be assumed that the positive direction is counter-clockwise; that is, opposite to the direction of rotation of the hands of a watch placed face upward in the plane of the paper.

28. Moment Represented by Area of Triangle.— If a triangle be constructed with its vertex at the origin of moments and its base in the position-line of a localized vector, the length of the base being numerically equal to the magnitude of the vector, the area of the triangle is numerically equal to half the moment of the vector. This follows immediately from the definition of moment.

29. Moment of Sum of Two Vectors.— If p, q and r represent three localized vectors such that $p + q = r$, and if their position-lines meet in a point, the algebraic sum of the moments of p and q with respect to any origin in their plane is equal to the moment of r with respect to that origin.

Let the three vectors p, q and r be represented in magnitude and direction by OA, OB and OC (Fig. 11) respectively, so that

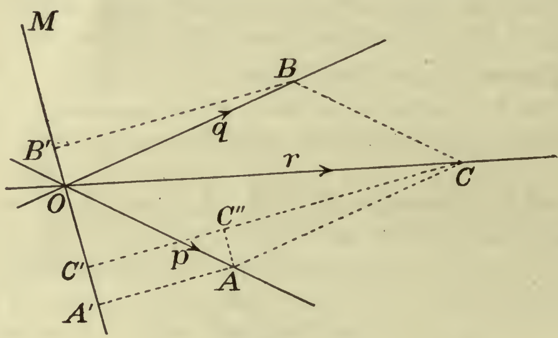


FIG. 11.

OC is the diagonal of the parallelogram of which OA and OB are sides. Let O be the point of intersection of the three position-lines, and M the origin of moments. Draw MO. Then

moment of $OA = 2$ (area of triangle MOA);
" " $OB = 2$ (" " " MOB);
" " $OC = 2$ (" " " MOC).

Draw AA', BB', CC', each perpendicular to MO. The double areas of the three triangles are respectively

$$MO \times AA',\ MO \times BB',\ MO \times CC'.$$

Now, $$CC' = AA' + BB'; \qquad . \quad . \quad . \quad (1)$$

for, drawing AC'' parallel to OM, it is seen that $AA' = C''C'$ and $BB' = CC''$. Hence

$$\text{area } MOA + \text{area } MOB = \text{area } MOC. \quad . \quad . \quad (2)$$

This proves the proposition for the case in which the moments of the two given vectors have the same sign.

The case in which the origin is so taken that the moments have opposite signs is represented in Fig. 12. The lettering is the same

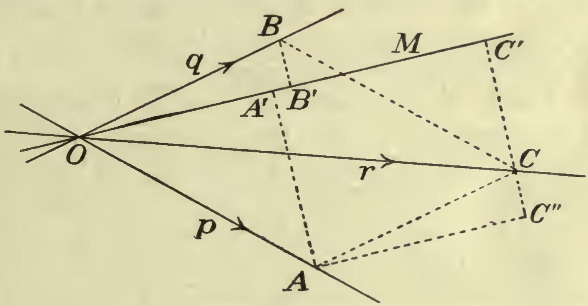


FIG. 12.

as in Fig. 11, and the reasoning to be employed is exactly similar. The algebraic sum of the moments is now the numerical difference. Thus, from the geometry of the figure, we have

$$CC' = AA' - BB' \qquad . \quad . \quad . \quad (1')$$

instead of equation (1), and therefore

$$\text{area } MOA - \text{area } MOB = \text{area } MOC, \quad . \quad (2')$$

instead of equation (2).

30. Vector Quantities in Mechanics.— Some of the important quantities dealt with in the science of Mechanics are vector quantities. The foregoing brief discussion is given as a useful introduction to the development of the elementary principles of Mechanics.

PART I. STATICS.

CHAPTER II.

FORCE AND STRESS.

§ 1. *Conception of Force.*

31. Motion of a Body if Uninfluenced by Other Bodies.— A body remains at rest, or moves in a straight line at a uniform rate, except in so far as its motion is influenced by other bodies. (Newton's first law of motion. See Art. 259.)

Since no body can be wholly free from the influence of other bodies, it is impossible to verify this law directly. The indirect evidence for its truth is, however, very strong. When a body is observed to depart from a condition of uniform rectilinear motion, it is always found reasonable to attribute this departure to the influence of other bodies. Thus, a stone thrown horizontally into the air departs from the condition of uniform motion in a straight line in two ways: its path curves downward, and its rate of motion changes. The first effect we attribute to the influence of the earth (which we say *attracts* the body), while the second we attribute partly to the "attraction" of the earth and partly to the influence of the air (which we say *resists* the motion of the body).

32. Force Defined.— When one body influences the motion of another body, the first is said to *exert a force* upon the second. Force may therefore be defined as follows:

A force is an action exerted by one body upon another, tending to change the state of motion of the body acted upon.

When an external body exerts a force upon any portion of the human body, the action (if sufficiently intense) is recognized by the senses. In such a case the name *push* or *pull* is given to the force. It is therefore natural to conceive of every force as a push or a pull. This conception will be found useful in deciding what may properly be called forces.

33. **Magnitude and Direction of a Force.**—When the action of forces is recognized by the senses, we form the idea that forces possess different magnitudes. Thus, if two pressures are successively applied to the same part of the body, one may be recognized as much greater than the other. We thus come to think of a force as possessing a definite magnitude ; although we are not able to compare the magnitudes of forces with precision by means of their immediate effects upon the senses.

Experience also gives the idea that a force possesses a definite direction ; but we are not able, from the immediate evidence of the senses, to compare the directions of forces with accuracy.

This conception of a force as possessing definite magnitude and direction is verified and made definite when the effects of forces upon the motions of bodies are studied.

34. **Force a Vector Quantity.**—Since a force possesses both a definite magnitude and a definite direction, it is a *vector quantity* (Art. 16).

Like other vector quantities, forces may be represented by directed right lines or vectors. To accomplish this, some length must be chosen to represent the unit force ; then any given force will be represented by a vector whose length is the same multiple of the chosen length that the magnitude of the force is of the unit force.

35. **Action and Reaction.**—If a body A exerts a force upon a second body B, the body B at the same time exerts upon A a force of equal magnitude in the opposite direction. One of the forces of such a pair is often called an action and the other a reaction ; and the principle may be stated in the words "to every action there corresponds an equal and contrary reaction." (Newton's third law of motion. See Art. 259.)

36. **Stress.**—A *stress* consists of two equal and opposite forces constituting the action and reaction between two bodies or portions of matter.

Illustrations.—Any two bodies attract each other in accordance with the law of gravitation ; the two forces they exert upon each other constitute a stress. Such a stress acts between the earth and the moon, or between any other two heavenly bodies.

Two electrified bodies attract (or repel) each other with equal and opposite forces, constituting a stress.

Two bodies in contact exert upon each other equal and opposite forces at the surface of touching; these forces constitute a stress.

37. Place of Application of a Force.— A force applied to a body may act either throughout a certain volume, or upon every part of a certain surface. Thus, the attraction of the earth upon a body according to the law of gravitation acts upon every part of the mass, and therefore its place of application is the whole space occupied by the body; while the pressure between two bodies which touch each other is distributed over their surface of contact. In both cases the forces are *distributed.*

A *concentrated* force is a force of finite magnitude applied at a point. Although this definition is ideal, since all forces are in reality distributed in their action, either throughout a volume or over an area, the conception of a concentrated force is a useful one. In many cases a relatively large force is applied throughout a small volume or over a small area, and the conception of a concentrated force is approximately realized.

By the study of the effects of forces upon the motions of the bodies to which they are applied, the conception is reached that a distributed force is "equivalent to" a concentrated force. It is with the laws of equivalence of systems of concentrated forces that the science of Statics mainly deals; a distributed force being regarded as the limiting case of a system of concentrated forces whose number becomes very great while their individual magnitudes become very small.

A force is thus regarded as having a definite *point of application* and a definite *line of action.* It is thus a *localized vector quantity* (Art. 26).

§ 2. *Classes of Forces.*

38. Conditions Under Which Forces Act.— The conditions under which bodies exert forces upon one another are known only from observation and experiment. It will be useful to enumerate a few cases of forces whose laws have been the subject of scientific investigation.

39. Examples of Forces.— (1) *Gravitation.*—Any two bodies exert upon each other attractive forces in accordance with the following law, called Newton's law of gravitation:

Every particle of matter attracts every other particle with a force which acts in the line joining the two particles, and whose magnitude is proportional directly to the product of their masses and inversely to the square of the distance between them.

(2) *Electrical forces.*—If two bodies are charged with electricity, each exerts a force upon the other. The forces are repulsive or attractive, according as the charges are "like" or "unlike" in kind.

(3) *Magnetic forces.*—Two magnets exert upon each other forces, both attractive and repulsive; like poles repelling and unlike attracting each other.

(4) *Electromagnetic forces.*—Forces are exerted upon each other by a magnet and a wire carrying an electric current; also by two wires carrying electric currents.

(5) *Molecular and atomic forces.*—The "atomic theory" of the constitution of matter assumes that a body of definite composition is made up of ultimate particles called *molecules*, each of which possesses the same physical properties as the whole body; while each molecule is made up of lesser parts called *atoms*, which do not separately possess the physical properties of the body. The theory assumes that forces (besides the universal force of gravitation) act between the ultimate particles.

40. Action at a Distance.—Forces are sometimes classified as "actions at a distance" and "actions by contact." Under the former would be classed gravitational, electrical, and magnetic forces, since they apparently are not exerted by means of any material connection between the bodies concerned. Modern researches have rendered it highly probable that electrical and magnetic forces are in reality actions transmitted through a "medium" which fills all space. Efforts have also been made to show that the force of gravitation may be explained in a similar manner. Whether this view be correct or not, the distinction between actions at a distance and actions by contact is practically useful.

41. Passive Resistances.—The forces which bodies exert upon one another are often divided into two classes, called respectively *active* forces and *passive resistances.* By an active force is meant one which acts independently of the state of motion of the body, and also independently of any other forces which may be applied to it; while a passive resistance comes into action only to prevent or to resist certain motions of the body.

Thus, if a body rests upon a horizontal table, it is acted upon by the attraction of the earth, which is an active force, tending to give it motion downward, and having the same magnitude and direction whatever other forces may be applied to the body; and also by a passive resistance, exerted upward by the table to resist the tendency of the body to move downward. This latter force acts only because the body has a tendency to move downward through the table. If, by means of the hand, there be applied to the body a supporting force of less magnitude than the downward pull of the earth, the resisting force exerted by the table is diminished, its amount being always just sufficient to neutralize the effect of the other forces. If the hand supports the body with an upward force equal to the downward attraction of the earth, the pressure of the table upon it becomes zero. If the upward force applied by the hand becomes still greater, the table exerts no force to resist the tendency of the body to rise. This illustrates the fact that a passive resistance can have any magnitude within certain limits, but that its value cannot go outside these limits.

If it were possible to analyze the forces exerted by one body upon another at their surface of contact, and by contiguous portions of the same body upon each other, in such a way as to determine the actions between the ultimate particles and the laws governing these molecular or atomic forces, it is probable that these would be found to have the same essential nature as the forces called active. But for practical purposes the distinction between active forces and passive resistances is often useful.

In the following Articles are considered two cases of passive resistances which are of importance in the discussion of the problems of Statics.

42. Pressure Between Bodies in Contact.— If two bodies touch each other, each exerts a force upon the other at the surface of contact. These contact forces are passive resistances, and (within certain limits) will take any values necessary to resist certain relative motions of the bodies. But the magnitudes and directions of the forces which the bodies can exert upon each other are subject to certain limitations depending upon the nature of the surfaces and the material composing the bodies.

Let A and B (Fig. 13) represent the two bodies. If for any reason A has a tendency to slide upon B, this tendency will be

resisted, and may be wholly neutralized. The rougher the surfaces of contact, the greater the force that can be exerted to resist the sliding. If the surfaces be very smooth, the possible magnitude of the force is very small. Hence we are led to the following definition :

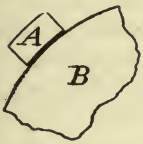


FIG. 13.

A *perfectly smooth surface* is one which can offer no resistance to the sliding of a body upon it.

Although this definition cannot be realized in the case of any actual body, certain surfaces approach near to the condition of perfect smoothness.

If for any reason A has a tendency to move toward B in the direction of the normal to their surface of contact, the latter body is able to exert a resisting force in the direction of the normal. The magnitude which this resisting force can take is limited only by the strength of the material of which B is composed. Such a "normal resistance" can be exerted by a smooth surface as well as by a rough one. The pressure exerted between rough surfaces will be considered later under the head of *friction.*

Smooth hinge.—A hinge joint is often used to connect two bodies, in such a way as to leave them free to move in a certain manner relatively to one another. Such a joint consists of a cylindrical pin which forms a part of one body, or is rigidly connected with it, and a cylindrical hole formed in the other body, into which the pin is inserted. Such a connection permits the bodies to turn relatively to each other about the axis of the pin, but (if the pin and hole are of equal diameter) prevents any other relative motion. (See Fig. 14.) In order to permit free motion about the axis, the pin must be slightly smaller than the hole, so that the two bodies are in contact along a straight element common to the two cylindrical surfaces. If these surfaces are smooth, the pressure between them has the direction of their common normal. In the problems of Statics, a smooth hinge is often introduced as a means of connecting bodies. In such a case the pressure between the bodies is to be taken as acting in a line through the center of the hinge.

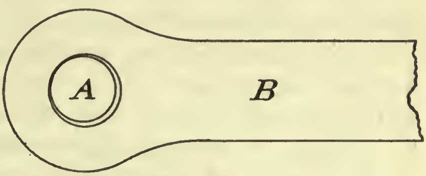


FIG. 14.

43. Tension in a Flexible Cord.— A *flexible cord* is one which may be bent. A *perfectly* flexible cord may be bent without resistance. No actual cord possesses perfect flexibility, but the resistance to bending may be very slight.

Let AB (Fig. 15) represent a portion of a perfectly flexible cord, and let two equal and opposite forces, P and P', be applied to it at A and B. Let C be any point between A and B. The force P tends to cause AC to move to the left; this is prevented by a force equal and opposite to P exerted at C by CB upon AC. Similarly, P' tends to cause CB to move to the right; and this is prevented by a force equal and opposite to P' exerted at C by AC upon CB. That is, the two parts of the cord exert upon each other at C equal and opposite forces, constituting a stress (Art. 36). The same is true at any other point between A and B. Such a stress (resisting a tendency of the two portions of the cord to separate in the direction of their length) is called a *tension*.

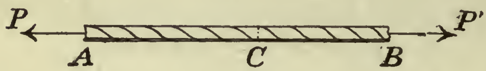
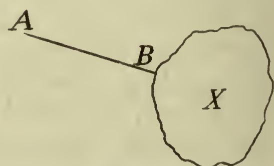


FIG. 15.

We have here illustrated a special case of the principle that *if a flexible cord is pulled tight between two points, any two adjacent portions exert upon each other forces parallel to the direction of the cord.*

This subject will be discussed more fully in a later chapter. It is introduced at this point because in the problems to be discussed in the following chapters a flexible cord is often used as a means of applying a force to a body. Thus, if a cord is attached to a body X (Fig. 16) at a point B, a force of any magnitude applied to the cord at another point A, in the direction BA, will cause the cord to exert an equal force upon the body at B.



FIF. 16.

§ 3. *Numerical Representation of Forces and of Masses — Gravitation System.*

44. Relation Between Units of Force and of Mass.— Although force and mass are two distinct kinds of quantity, so that it is possible to choose the unit in which each is expressed independently of

the other, yet it is convenient to choose these units so that one depends upon the other. The method of choosing units which will now be described makes the unit mass arbitrary, while the unit of force depends upon that of mass. This is the *gravitation system* of units. In a later chapter another system will be described.

45. **Unit Mass.**— The unit mass is taken as the quantity of matter in a certain piece of platinum arbitrarily chosen and established as the standard by act of the British Parliament. This unit mass is called *one pound.*

46. **Weight.**—The *weight* of a body is the force with which the earth attracts it, in accordance with the law of gravitation. From this law (Art. 39) it follows that the weights of bodies in the same locality at the surface of the earth are proportional to their masses. But if the bodies are in different positions on the earth, or at different elevations above the surface, their weights may not be proportional to their masses.

In problems relating to bodies near the earth, the weight of a body has often to be included among the forces applied to it. By the law of gravitation every particle of the earth attracts every particle of the body. These several forces may for many purposes be regarded as equivalent to a single force applied at a certain point called the *center of gravity* of the body. The truth of this statement is for the present assumed without proof. (See Chapter IX.)

47. **Unit Force.**— The *gravitation unit of force* is a force equal to the weight of a unit mass at the earth's surface.

A *pound force* is a force equal to the weight of a pound mass at the earth's surface.

The pound force as thus defined has not the same value for all positions on the surface of the earth, since the weight of any given body varies if it be taken to different localities. The variations are, however, comparatively small, and for many purposes unimportant. The pound force may be made wholly definite by specifying a certain place at which its value is to be determined.

It is unfortunate that the word pound is used in two senses, applying both to a unit force and to a unit mass. The usage is, however, so common that it is important for the student to become familiar with it. The double meaning arises from the fact that the

most convenient as well as the most accurate method of comparing the masses of bodies is by determining the ratio of their weights.

48. **Comparison of Forces.**—The magnitude of a force in terms of the gravitation unit, or pound force, may be found by determining how many pounds mass it will support against the attraction of the earth. This is the method employed, directly or indirectly, in many machines for testing the strength of materials.

49. **Measurement of Masses.**—The mass of a body may be determined by comparing its *weight* with that of a body or bodies of known mass. This is the method adopted in "weighing" a body on an ordinary balance consisting of a lever or a system of levers. The weight of a body as thus determined is a correct indication of its mass, provided the masses of the standard bodies or "weights" are accurately known; for if the same experiment be conducted in two different localities, the weights of the standard bodies will change in the same ratio as the weights of the bodies balanced against them.

If the weight of a body is determined by a spring balance, the result may be different if the weighing is done in different localities; for in this case the process of weighing consists, not in comparing the weight of one body with that of another, but in determining directly the pull exerted by the earth * upon the body by ascertaining how much this pull will stretch a spring. If the weight of the body changes, therefore, as it may if its location is changed, this change will be indicated by a corresponding variation in the amount of stretching of the spring.

50. **Metric System.**—In the gravitation system of estimating forces and masses, the unit mass is wholly arbitrary. Besides the units above described, which owe their establishment to British law and custom, there is another important set, based upon the French *gram* or the *kilogram*. A gram is very nearly equal to the mass of a cubic centimeter of pure water at the temperature of maximum density. This relation is not exact, however, and the gram, like the pound, must be regarded as an arbitrary unit, whose value is to be

---

* What is actually measured is the "apparent" pull of the earth upon the body, which differs from the true attractive force for reasons which cannot be here discussed, the chief of them being the diurnal rotation of the earth. See Art. 311.

determined by reference to a certain standard body. For ordinary purposes the kilogram, equal to 1,000 grams, is often more convenient than the gram.

The gravitation unit of force, in the French or metric system, may be taken as the weight of a body of one kilogram mass. For brevity this unit may be called a force of one kilogram. To make the unit definite, the position must be specified, since the weight of the same body is different in different places.

The following is the relation between the pound and the kilogram:

1 kilogram = 2.2046 pounds.
1 pound = 0.45359 kilogram.

The gravitation system of units of force and mass, although the most convenient in many practical applications, is not the best for the purposes of pure science. The discussion of other systems must, however, be deferred.

§ 4. *Definitions.*

51. **Concurrent and Non-concurrent Forces.**— Forces acting upon a body are *concurrent* when they have the same point of application. When applied at different points they are *non-concurrent.*

52. **Coplanar Forces** are those whose lines of action lie in the same plane. The following pages treat mainly of coplanar systems.

53. **A Couple** is a system consisting of two forces, equal in magnitude, opposite in direction, and having different lines of action. The perpendicular distance between the two lines of action is called the *arm* of the couple.

54. **Equivalent Systems of Forces.**— Two systems of forces are *equivalent* if either may be substituted for the other without change of effect.

From this definition it follows that two systems, each of which is equivalent to a third system, are equivalent to each other.

55. **Resultant.**— A single force that is equivalent to a given system of forces is called the *resultant* of that system. It will be shown subsequently that a system of forces may not be equivalent to any

single force. When such is the case, the simplest system equivalent to the given system may be called its resultant.

Any forces having a given force for their resultant are called *components* of that force.

56. Composition and Resolution of Forces.— Having given any system of forces, the process of finding an equivalent system is called the *composition of forces* if the system determined is simpler than the given system; if the reverse is the case, the process is called the *resolution of forces*.

The process of finding the resultant of any given forces is the most important case of composition; while the process of finding two or more forces, which together are equivalent to a single given force, is the most common case of resolution.

57. Equilibrium.— A system of forces applied to a body is in *equilibrium* if the state of motion of the body is not changed by their action.

The term equilibrium is also applied to the *body* upon which the forces act; a body is said to be in equilibrium if its state of motion is unchanged by the action of all applied forces.

CHAPTER III.

CONCURRENT FORCES.

§ 1. *Composition and Resolution of Concurrent Forces.*

58. **Forces Having the Same Line of Action.**— The resultant of two concurrent forces acting in the same direction is a force in that direction equal to their sum.

The resultant of two concurrent forces acting in opposite directions is a force whose magnitude is equal to the difference between the magnitudes of the given forces, and whose direction is that of the greater.

No proof need be given of the truth of these statements. They must be regarded as following immediately from our conception of force and from our experience regarding the effects of forces. They are special cases of the general principle of the parallelogram of forces (Art. 59).

These principles may be generalized in the following manner :

(1) The resultant of any number of concurrent forces having the same direction is a force in that direction whose magnitude is equal to the sum of the magnitudes of the given forces.

(2) The resultant of any concurrent forces having the same line of action is found by combining those having one direction into a single force and those having the opposite direction into another force, and then combining these partial resultants according to the rule already stated.

If signs plus and minus are used to distinguish the two opposite directions along the common line of action of the forces, the foregoing principles may be included in the following general rule:

The resultant of any concurrent forces having the same line of action is a force equal to their algebraic sum.

59. **Resultant of Any Two Concurrent Forces.**— If any two concurrent forces be represented in magnitude and direction by lines drawn from the same point, their resultant is represented in magnitude and direction by the diagonal (drawn from that point) of the parallelogram of which the two lines are adjacent sides.

Thus let OA and OB (Fig. 17) represent the two given forces, and OC the diagonal of the parallelogram of which OA and OB are sides; then OC represents the resultant.

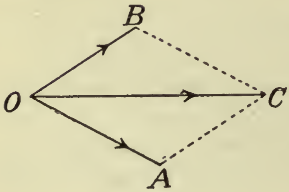


FIG. 17.

This principle is known as the *parallelogram of forces.* Many proofs of the proposition have been offered, but none of these has been generally accepted as satisfactory. Perhaps the most satisfactory view is that which regards this principle as one of the fundamental laws of force, depending for its verification upon experience. The matter is further considered in Chapter XIV.

60. Triangle of Forces.— Since the opposite sides of a parallelogram are equal, the magnitude and direction of the resultant of two forces may be found by constructing a triangle instead of a parallelogram. Thus, any two forces being given, let them be represented in magnitude and direction by lines OA and AC laid off consecutively (Fig. 17); then OC represents the magnitude and direction of the resultant. For, if OB be drawn equal and parallel to AC, OC is a diagonal of the parallelogram of which OA and OB are adjacent sides. The lines OB, BC are not needed for the determination of OC. Evidently, the principle of the parallelogram of forces is equivalent to the following:

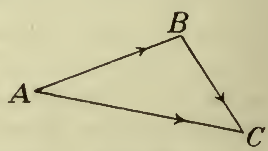


FIG. 18.

If A, B, C are three points so chosen that AB, BC represent, in magnitude and direction, two concurrent forces, their resultant is represented in magnitude and direction by AC.

This principle is known as the *triangle of forces.*

The point of application of the resultant is the same as that of the components.

61. Computation of Resultant of Two Concurrent Forces.— Let P and Q represent the magnitudes of two concurrent forces, and θ the angle between their lines of action. [The angle is to be measured between the positive directions of the lines of action. Thus, if XX' and YY' (Fig. 19) are the lines of action, the directions

being as indicated by the arrows, θ is the angle $X'OY'$.] Laying off AB, BC to represent P and Q, AC will represent the resultant R. By elementary Trigonometry, the magnitude of the resultant is given by the formula

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta.$$

To find the direction of the resultant, let angle $CAB = \alpha$, angle $ACB = \beta$; then

$$\frac{\sin\alpha}{Q} = \frac{\sin\beta}{P} = \frac{\sin\theta}{R};$$

or

$$\sin\alpha = \frac{Q}{R}\sin\theta; \quad \sin\beta = \frac{P}{R}\sin\theta.$$

EXAMPLES.

1. A particle is acted upon by forces of 10 lbs. and 20 lbs. whose lines of action include an angle of 25°. Determine the magnitude and direction of a single force which would produce the same effect.

Ans. A force of 29.4 lbs. making an angle of 16° 42′ with the force of 10 lbs.

2. Two men pull a body horizontally by means of ropes. One exerts a force of 28 lbs. directly north, the other a force of 42 lbs. directed N. 42° E. What single force would be equivalent to the two?

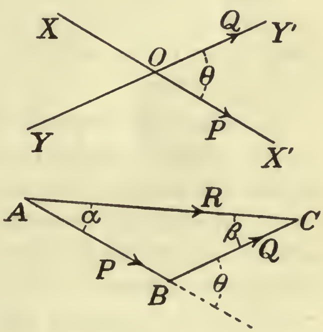


FIG. 19.

3. Two persons lifting a body exert forces of 44 lbs. and 60 lbs. in directions inclined 28° to the vertical on opposite sides. What single force would produce the same effect?

Ans. A force of 92.1 lbs. at angle 32° 40′ with force of 44 lbs.

62. Resultant of Any Number of Concurrent Forces.— The resultant of any number of concurrent forces may be found by first finding the resultant of two of them, then combining this resultant with a third force, and so on.

Thus, if AB and BC (Fig. 20) represent two of the forces in magnitude and direction, their resultant is represented by AC. Draw CD to represent a third force; the resultant of AC and CD is AD, which is therefore the resultant of the three forces represented by

AB, BC, and CD. If DE represents a fourth force, the resultant of AD and DE is AE, which is therefore the resultant of AB, BC, CD and DE. This process may be extended to include any number of forces. Evidently the lines AC and AD are not needed in the construction.

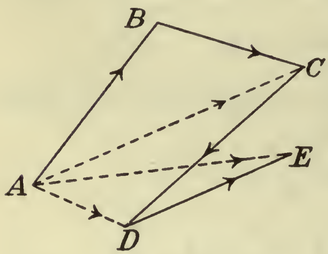


FIG. 20.

The above process may be described as follows:

If any number of concurrent forces are represented in magnitude and direction by lines forming sides of a continuously described polygon, the line drawn from the starting point to close the polygon represents the resultant in magnitude and direction.

This principle is called the *polygon of forces.*

The sides of the polygon cannot, of course, represent the lines of action of the forces, since these all intersect in the point at which the forces are applied.

63. Resultant as Vector Sum.—The process of finding the resultant of several forces by constructing the polygon is evidently the same as the process of vector addition (Art. 18). That is,

The resultant of any number of concurrent forces is a force equal to their vector sum, its point of application being the same as that of the given forces.

64. Vector Diagrams and Space Diagrams.—In combining forces by geometrical construction, they must be represented in magnitude and direction, and also in line of action. In most cases it is desirable, in order to prevent confusion, to draw two separate diagrams, one showing the lines of action of the forces, the other representing them in magnitude and direction only. These two classes of diagrams may be called *space diagrams* and *vector diagrams* respectively.

65. Computation of Resultant by Force Polygon.—The resultant of any coplanar concurrent forces may be computed by graphical construction, by drawing the force polygon to scale and measuring the line representing the resultant. Or, the length and direction of the closing side of the force polygon may be computed

trigonometrically from the force polygon. A more convenient trigonometric method will be given later.

66. Resolution of a Force Into Any Number of Components.— A force may be resolved into any number of components by means of the polygon of forces. If the magnitude and direction of the given force be represented by a line, this line may be made one side of a polygon, and the other sides will represent, in magnitude and direction, forces of which the given force is the resultant. Thus, let the given force be represented by the vector AB (Fig. 21). Choose any number of points (as C, D, E), and draw lines forming the polygon $ACDEB$. Then AC, CD, DE, EB represent, in magnitude and direction, four forces whose resultant is AB.

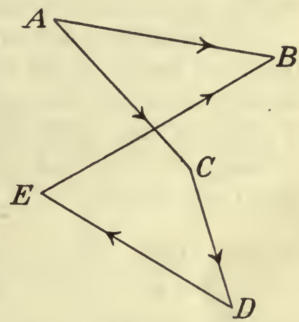


FIG. 21.

A force may obviously be replaced by other forces in an infinite number of ways. Some special cases of the general problem of resolution will now be considered.

67. Resolution of a Force Into Two Components.— This is a special case of the general problem of resolution into any number of components. If a force be represented in magnitude and direction by a vector AB, and if C be any point whatever, the vectors AC, CB represent two forces which together are equivalent to the given force. Since an infinite number of triangles may be drawn, each having any given line as one side, it follows that a force may be replaced by two forces in an infinite number of ways.

To make the problem determinate, certain conditions must be specified which the components are to satisfy. The following cases furnish determinate problems. Each should be solved both by geometrical construction and by algebraic computation.

(1) Let a force be resolved into two components whose lines of action are given.

SOLUTION. (*a*) *Geometrical.*— Let AB (Fig. 22) represent the given force in magnitude and direction, and let the two components have lines of action parallel to XX and YY. From A draw a line parallel to XX, and from B a line parallel to YY, C being the point of intersection of these two lines; then AC and CB represent

the required components in magnitude and direction. The construction may evidently be varied by drawing from A a line parallel to YY and from B a line parallel to XX, giving AC' and $C'B$ as the two components. The result is the same as before, since AC and $C'B$ are equal vectors, as are also AC' and CB.

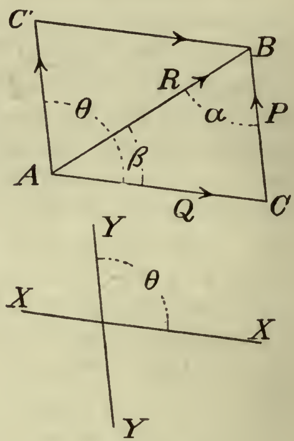


FIG. 22.

(*b*) *Algebraic.*—From the given data there are known in the triangle ACB the side AB and all the angles. Hence, if $AB = R$, $AC = Q$, $CB = P$, $ACB = 180° - \theta$, $ABC = \alpha$, $BAC = \beta$, we have (as in Art. 61)

$$\frac{Q}{\sin \alpha} = \frac{P}{\sin \beta} = \frac{R}{\sin \theta},$$

from which P and Q may be computed.

(2) Let the two components be given in magnitude only.

(3) Let the magnitude of one and the direction of the other be given.

(4) Let one component be wholly given.

EXAMPLES.

1. A force of 50 lbs. is equivalent to two forces whose directions are inclined to that of the given force at angles of 12° and 86° respectively; determine their magnitudes. *Ans.* 50.6 lbs. and 10.5 lbs.

2. A force of 83 lbs. is to be replaced by two components, one of which is a force of 36 lbs. at right angles to the given force. Determine the magnitude and direction of the other component.

3. Resolve a force of 200 lbs. into two components, of 130 lbs. and 98 lbs. respectively. Determine the directions of the two components.

Ans. Angle between resultant and greater component = 24° 34′.

4. A force of 150 lbs. is to be resolved into two components, one acting at an angle of 20° with the given force, the other having a magnitude of 80 lbs. Determine completely the two components.

5. A force of 150 lbs. is to be resolved into two components, one acting at an angle of 20° with the given force. What is the least possible magnitude of the other? *Ans.* 51.31 lbs.

68. Resolved Part of a Force.—*Definition.*—If a force is equivalent to two components at right angles to each other, each is called a *resolved part* of the given force.

The resolved part of a force represented by AB (Fig. 23) in the direction of a line MN is represented in magnitude and direction by $A'B'$, the orthographic projection of AB upon MN.

Proposition.—The resolved part, in a given direction, of the resultant of any concurrent forces, is equal to the algebraic sum of the resolved parts of the components in that direction.

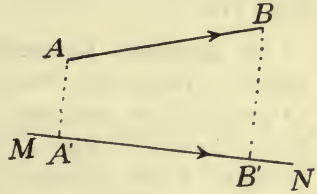


FIG. 23.

Since the resultant is the vector sum of the components, this proposition is a special case of that stated in Art. 25.

Algebraic expression for the resolved part.—From the definition it follows that the resolved part of a force of magnitude P in a direction making an angle θ with that of the force is equal to $P \cos \theta$. If θ lies between 90° and 270° its cosine is negative. Hence if θ is measured from the positive direction of the line along which the resolution is made, the product $P \cos \theta$ gives the resolved part with proper sign, whatever the value of the angle.

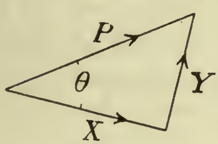


FIG. 24.

Let X and Y be the resolved parts of a force P in two directions at right angles to each other, θ being the angle between P and X, and $\pi/2 - \theta$ the angle between P and Y. Then

$$X = P \cos \theta;$$

$$Y = P \cos (\pi/2 - \theta) = P \sin \theta;$$

$$P^2 = X^2 + Y^2;$$

$$\cos \theta = \frac{X}{P}; \quad \sin \theta = \frac{Y}{P}.$$

From these equations we may compute the resolved parts when the force and the directions of resolution are given; or we may compute the magnitude and direction of the force when its resolved parts in two directions at right angles to each other are known.

EXAMPLES.

1. Let 20 lbs. and 40 lbs. be the resolved parts of a force in two directions at right angles to each other. Determine the magnitude and direction of the force.

2. Compute the resolved part of a force of 235 lbs. in a direction inclined 25° to that of the force; also in a direction inclined 160° to that of the force.

69. Algebraic Computation of Resultant of Any Concurrent Forces.— From the principle of the triangle of forces, it is possible to compute the magnitude and direction of the resultant of any known concurrent forces. Thus, the resultant of any two may be computed by the solution of a triangle; this resultant may be combined with a third force and their resultant computed in a similar manner; and the process may be continued until the resultant of the whole system is determined. This process would, however, be very laborious, and in most cases the following method is to be preferred.

Choose a pair of rectangular axes in the plane of the forces, and let the given forces be P_1, making angles α_1, β_1 with the axes; P_2, making angles α_2, β_2 with the axes; etc. Replace each force by its resolved parts parallel to the two axes; P_1 being replaced by components $P_1 \cos \alpha_1$, $P_1 \cos \beta_1$; P_2 by components $P_2 \cos \alpha_2$, $P_2 \cos \beta_2$; etc. The given forces are thus replaced by two systems of collinear forces. The resultants of these two systems are respectively

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \quad . \quad . \quad . \quad = X,$$

and

$$P_1 \cos \beta_1 + P_2 \cos \beta_2 + \quad . \quad . \quad . \quad = Y.$$

The resultant of the whole system is equal to the resultant of X and Y. Let R denote this resultant and a, b the angles it makes with the x- and y-axes respectively; then

$$R^2 = X^2 + Y^2;$$

$$\cos a = \frac{X}{R}; \quad \cos b = \frac{Y}{R}.$$

EXAMPLES.

1. Find the resultant of the following concurrent forces: 23 lbs. directed N. 40° E.; 42 lbs. S. 20° E.; 86 lbs. due E.; 56 lbs. N. 33° W.

2. A body at a point O is pulled equally by three men in directions OA, OB and OC, such that $AOB = BOC = 60°$ and $AOC = 120°$. What single force would produce the same effect?

Ans. If $P =$ pull exerted by one man, the resultant is a force $2P$ in the direction OB.

3. A body of 145 lbs. mass is acted upon by a horizontal force of 63 lbs. What single force would be equivalent to the horizontal force and the weight of the body?

§ 2. *Moments of Concurrent Forces.*

70. Moment of a Force.— The moment of a force with respect to a point is the product of the magnitude of the force into the perpendicular distance of its line of action from the given point.

This is a particular case of the definition of the moment of a localized vector, already given (Art. 27). The sign of the moment of a force will usually be assumed to follow the convention there adopted.

Moment about an axis.— A line through the origin of moments, perpendicular to the plane containing the origin and the line of action of the force, may be called an *axis of moments*. The moment of the force with respect to the origin may also be regarded as its moment with respect to this axis.

71. Moment of Resultant of Two Concurrent Forces.— The moment of the resultant of two concurrent forces with reference to a point in their plane is equal to the algebraic sum of their separate moments with reference to that point.

Since the resultant of two concurrent forces is a force equal to their vector sum applied at their common point of application, this proposition is a particular case of that proved in Art. 29 for localized vectors. It was there assumed that the two vectors were not parallel. If, however, they are parallel and applied at the same point, the truth of the proposition is evident.

72. Moment of Resultant of Any Number of Concurrent Forces.— The moment of the resultant of any number of concurrent forces with reference to any point in their plane is equal to the algebraic sum of their separate moments with reference to that point.

The proposition having been proved for two forces, let the resultant of any two of the given forces be combined with a third; since the proposition is true for these two forces and their resultant, it is true

for the three forces and their resultant. Similarly, it is true for this resultant and a fourth force, and is therefore true for the four forces and their resultant. The same line of reasoning may be extended to include any number of forces.

EXAMPLES.

1. Compute the moments of the four forces described in Ex. 1, Art. 69, the origin of moments being a point 8 ft. directly north from the point of application of the forces.

2. Determine the line of action of the resultant of two forces of 18 lbs. and 12 lbs. whose lines of action include an angle of 60°. Compute the moments of the two forces with respect to an origin on the action-line of the resultant, 4 ft. from the point of application of the forces. Test the truth of the above proposition.

3. In Ex. 2, compute the moments of the two forces and their resultant with respect to an origin on the line of action of the force of 12 lbs., 5 ft. from the point of application.

Ans. 77.94 ft.-lbs; 0; 77.94 ft.-lbs.

4. Prove that the moments of two concurrent forces are equal in magnitude for any origin on the line of action of the resultant.

§ 3. *Equilibrium of Concurrent Forces.*

73. General Condition of Equilibrium.— From the definitions of equilibrium (Art. 57) and of resultant (Art. 55) it follows immediately that *if a system of forces is in equilibrium the resultant is zero;* and conversely, *if the resultant is zero the system is in equilibrium.*

This is the *general* condition of equilibrium. From it may be derived various special conditions, adapted to the discussion of different classes of problems.

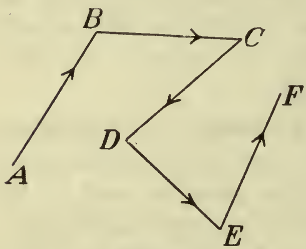


FIG. 25.

74. Geometrical Condition of Equilibrium.—*Force polygon.*—The figure formed by drawing in succession vectors representing any number of forces in magnitude and direction is called a force polygon for those forces. Thus, Fig. 25 shows a force polygon for five forces represented by the vectors AB, BC, CD, DE, EF.

In general the initial and final points (as A and F, Fig. 25) do not coincide. In case they do coincide the polygon is said to *close.* The order in which the vectors representing the forces are drawn does not affect the relative positions of the initial and final points.

Condition of equilibrium.—If A and F are the initial and final points of a force polygon drawn for any given forces, AF represents their resultant in magnitude and direction. Hence the resultant will be zero if F coincides with A, but not otherwise. Therefore,

If any number of concurrent forces form a system in equilibrium, their force polygon is closed. And conversely, *if the force polygon closes, the system is in equilibrium.*

75. Moment-Condition of Equilibrium.—*Proposition.*—If any number of concurrent forces are in equilibrium, the algebraic sum of their moments with respect to any point in their plane is zero.

It has been shown that the moment of the resultant, with respect to any origin, is equal to the algebraic sum of the moments of the components with respect to that origin. If the system is in equilibrium the resultant is zero, therefore its moment is zero whatever the origin; which proves the proposition.

76. Equations of Equilibrium.—From the principles stated in the last two Articles, any number of equations may be written which must be satisfied by a system of concurrent forces in equilibrium. These equations are of two kinds:

(a) The sum of the resolved parts of the given forces in any direction must be zero.

For, since the force polygon must close for equilibrium, the algebraic sum of the projections of its sides upon any line must equal zero.

(b) The sum of the moments must be zero for any origin.

Although each of these two general conditions leads to an infinite number of equations, only two of these can be independent. This will be seen by considering what any one equation implies.

(1) If the sum of the resolved parts in any direction is zero, the resultant force, if one exists, must be perpendicular to that direction.

(2) If the sum of the moments is zero with respect to any point, the resultant force, if one exists, must act in a line passing through that point.

It is now evident that there will be equilibrium if either of the following conditions is satisfied:

I. The sum of the resolved parts of the forces is zero in each of two directions.

II. The sum of the moments is zero with respect to each of two points not collinear with the point of application of the forces.

III. The sum of the moments is zero for one origin (not coinciding with the point of application of the forces), and the sum of the resolved parts is zero for any one direction not perpendicular to the line joining the origin of moments with the point of application.

77. Algebraic Deduction of Conditions of Equilibrium.— With the notation of Art. 69, we have

$$X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \quad . \quad . \quad . \quad ,$$

$$Y = P_1 \cos \beta_1 + P_2 \cos \beta_2 + \quad . \quad . \quad . \quad ,$$

$$R = \sqrt{X^2 + Y^2}.$$

The general condition of equilibrium, $R = 0$, requires that

$$X = 0 \text{ and } Y = 0.$$

For unless X^2 and Y^2 are separately equal to zero, their sum cannot equal zero unless one of them is negative; but this would make X or Y imaginary. It follows that the condition $R = 0$ is equivalent to the two conditions

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \quad . \quad . \quad . \quad = 0; \qquad (1)$$

$$P_1 \cos \beta_1 + P_2 \cos \beta_2 + \quad . \quad . \quad . \quad = 0. \qquad (2)$$

Since the axes along which the forces are resolved may be any pair of rectangular axes in the plane of the forces, it follows that for equilibrium the sum of the resolved parts of the forces in any direction must be zero. This agrees with Art. 76.

It might be shown algebraically that if two equations such as (1) and (2) are satisfied, every equation obtained by resolving the forces in any direction or by taking their moments about any origin must also be satisfied.

78. Solution of Problems in Equilibrium.— In solving problems in equilibrium algebraically, it is necessary to write as many independent equations as there are unknown quantities to be determined. From the above principles of equilibrium for concurrent forces, two independent equations can always be written. If the

number of unknown quantities is greater than two, the principles of equilibrium alone are not sufficient for their determination. In such a case the problem, if determinate, involves something more than the principles of equilibrium.

From the preceding discussion it is evident that the two independent *statical* equations can always be written with certainty. If the solution of the problem presents difficulty it lies usually in the algebraic solution of the equations.

In writing the two independent equations of equilibrium in either of the three allowable ways (Art. 76), simplification is usually possible by taking advantage of the following obvious principles:

(*a*) If the resolution is made in a direction perpendicular to a force, that force does not enter the resulting equation.

(*b*) If moments are taken about an origin lying on the line of action of a force, that force does not enter the resulting equation.

In many cases the "geometrical condition of equilibrium," that the force polygon must close, leads to a short solution without the use of the equations of equilibrium.

It is thus seen that any problem may be analyzed in two ways,—geometrically and algebraically. In solving the examples that follow, the student is advised to use both methods. The geometrical method frequently has the advantage of giving a rapid solution, while it also gives a clear view of the relations of the forces. On the other hand, the algebraic method has the advantage of affording an exhaustive treatment applicable to any determinate problem.

79. Applications.—The methods of applying the foregoing principles will be illustrated by the solution of the following problems.

I. A body of 40 lbs. mass is suspended by a cord from a ring which is supported by two cords making angles of 30° and 70° with the vertical. Determine the tensions in the cords.*

Two systems of forces in equilibrium are presented here,—(*a*) two forces acting upon the suspended body (its weight of 40 lbs. directed downward and an upward force exerted by the string), and (*b*) three forces acting upon the ring (exerted by the three cords). The conditions of equilibrium are to be applied to each system separately.

---

*In all cases in which cords are introduced they are understood to be perfectly flexible. (Art. 43.)

(*a*) The upward force exerted by the cord CD upon the body D (Fig. 26) must equal the downward force of 40 lbs. due to gravity.

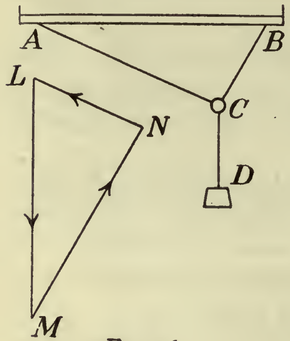


FIG. 26.

(*b*) The cord CD must pull downward upon the ring C with a force equal to its upward pull upon D. Hence, of the three forces acting upon the ring, one is known completely and the directions of the other two are known.

Geometrical solution.— The vectors representing the three forces applied to the ring must form a closed triangle. Draw LM (Fig. 26) vertically downward to represent the tension in CD, MN parallel to CB, LN parallel to AC; then MN and NL must represent the forces exerted upon the ring by the cords BC and AC respectively.

From the given data, $LM = 40$ lbs., angle $LMN = 30°$, angle $MLN = 70°$, angle $LNM = 80°$. Hence, by trigonometry,

$$\frac{LN}{\sin 30°} = \frac{MN}{\sin 70°} = \frac{LM}{\sin 80°},$$

or $$LN = LM\frac{\sin 30°}{\sin 80°} = \frac{0.5}{0.9848} \times 40 \text{ lbs.} = 20.3 \text{ lbs.};$$

$$MN = LM\frac{\sin 70°}{\sin 80°} = \frac{0.9397}{0.9848} \times 40 \text{ lbs.} = 38.2 \text{ lbs.}$$

Algebraic solution.— Let P and Q denote the tensions in AC and CB, and let the independent equations of equilibrium be formed by resolving forces horizontally and vertically. Then (taking upward and toward the right as positive directions)

$$-P\cos 20° + Q\cos 60° = 0;$$

$$P\cos 70° + Q\cos 30° - 40 \text{ lbs.} = 0.$$

Or $$-0.9397\,P + 0.5000\,Q = 0;$$

$$0.3420\,P + 0.8660\,Q = 40 \text{ lbs.}$$

Solving, $$P = 20.3 \text{ lbs.},\ Q = 38.2 \text{ lbs.}$$

II. A body of 60 lbs. mass rests against a smooth plane surface inclined 25° to the horizontal. It is supported partly by the plane and partly by a cord inclined 20° to the plane. What are the magnitudes of the supporting forces? (Fig. 27.)

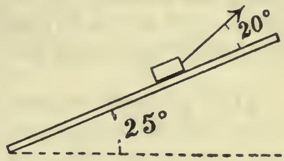


FIG. 27.

The forces acting upon the body are three, all known in direction (the weight acting vertically downward, the pull of the cord along its length, and the pressure of the smooth plane normally to its surface), and one known in magnitude.

Geometrical solution.— The triangle of forces can be drawn and the unknown forces determined as in the preceding problem.

Algebraic solution.— In writing the equations of equilibrium, it will be advantageous to resolve forces parallel and perpendicular to the plane. This gives an equation not involving the normal pressure.

Let $P =$ tension in cord, and $Q =$ pressure exerted by plane.

Resolving parallel to the plane,

$$P \cos 20^\circ - 60 \cos 65^\circ = 0.$$

Resolving perpendicularly to the plane,

$$P \cos 70^\circ + Q - 60 \cos 25^\circ = 0.$$

Solving, $P = 27.0$ lbs., $Q = 45.1$ lbs.

EXAMPLES.

1. A body of 20 lbs. mass resting upon a horizontal surface is acted upon by gravity and by a horizontal force of 5 lbs. The supporting body exerts upon it such a force as to hold it at rest. Determine the magnitude and direction of this force.

Ans. 20.6 lbs., inclined 14° 2′ to the normal.

2. A body of 120 lbs. mass, suspended by a flexible string, is pulled horizontally with a force of 45 lbs. Determine the direction of the suspending cord and the tension sustained by it. (Fig. 28.)

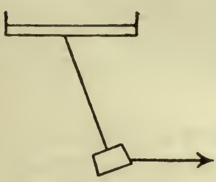


FIG. 28.

3. A body of 70 lbs. mass is suspended by a flexible cord from a ring which is supported by two cords making angles of 25° and 60° respectively with the vertical. Determine the tensions in the strings.

4. A body of P lbs. mass is suspended by

a cord from a ring which is supported by two cords making angles α and β with the vertical. Determine the tensions in the cords.

Ans. $P \sin \alpha / \sin (\alpha + \beta)$ and $P \sin \beta / \sin (\alpha + \beta)$.

5. A body of 70 lbs. mass is suspended as in Ex. 3, and is also pulled horizontally by means of an attached cord with a force of 10 lbs. Determine the tension in each cord. (Fig. 29.)

[This presents two sets of concurrent forces, one set acting on the suspended body, the other acting on the ring.]

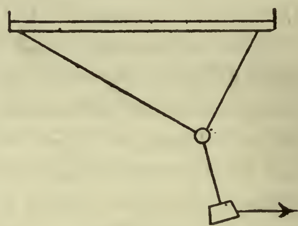


FIG. 29.

6. If the horizontal force in Ex. 5 be gradually increased, one of the cords will finally cease to act. For what value of the horizontal force will this occur? In this limiting case, what tensions are sustained by the cords?

Ans. Horizontal force = 121.24 lbs. Tensions = 140 lbs. and 0.

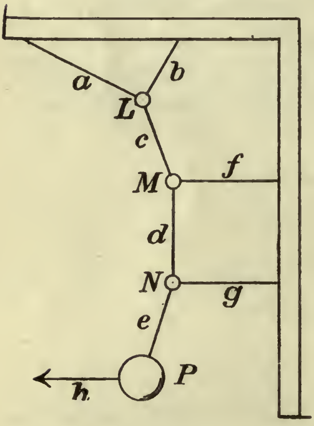


FIG. 30.

7. In Fig. 30 the cords a, b, c and d are inclined to the vertical at angles of 70°, 30°, 20° and 0° respectively, while f, g and h are horizontal. The mass of the body P is 60 lbs., and the horizontal force exerted by the cord h is 20 lbs. Determine the tension in each of the cords.

[Apply conditions of equilibrium separately to the forces concurrent at L, at M, at N and at P, beginning with P.]

Ans. Tension in a = 49.7 lbs.; in c, 63.8 lbs.; in g, 20 lbs.

80. Equilibrium of a Particle on a Smooth Surface. — If a particle rests against a smooth surface of any form, the pressure exerted upon the particle by the surface is a "passive resistance" (Art. 41) whose direction is known, being that of the normal to the surface. If the surface is a plane, the direction of the pressure is the same whatever the position of the particle; but for a curved surface the direction of the pressure exerted by the surface depends upon the position.

To solve a problem relating to the equilibrium of a particle on a smooth surface, the general equations of equilibrium may be written in the usual manner; but among the forces to be included in the

system is the unknown pressure due to the surface. The direction of this reaction is known if the position of the particle is known; otherwise it must be expressed in terms of the variable coördinates of the surface. The consideration of this problem will be confined to the case in which the particle and the applied forces are restricted to a plane curve lying in the given surface.

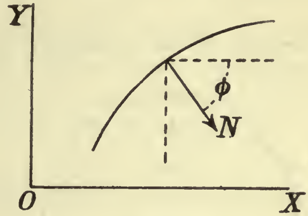


FIG. 31.

Let the particle be restricted to the plane curve shown in Fig. 31, and let the equation of this curve referred to rectangular axes OX, OY be known. If N denotes the magnitude of the unknown normal pressure acting on the particle, and ϕ the angle between the x-axis and the direction of N, we have

$$x\text{-component of } N = N \cos \phi;$$

$$y\text{-component of } N = -N \sin \phi.$$

But since N has the direction of the normal to the curve,

$$\cos \phi = \frac{dy}{ds} = \frac{dy/dx}{\sqrt{1 + (dy/dx)^2}};$$

$$\sin \phi = \frac{dx}{ds} = \frac{1}{\sqrt{1 + (dy/dx)^2}}.$$

The value of dy/dx in terms of x and y can be found from the equation of the curve.

In many cases the direction-angles of N can be expressed more conveniently in some other manner.

The method of solving problems of this kind will now be illustrated.

I. A smooth wire bent into the form of a circle and placed with a diameter vertical carries a bead of W lbs. mass, to which is applied a horizontal force of P lbs. In what position can the bead be in equilibrium?

Algebraic solution.— Let the radius drawn to the bead make with the vertical an angle θ, and let N denote the pressure of the wire upon the bead, its direction being radial. (Fig. 32.)

Resolving forces horizontally,

$$P - N \sin \theta = 0.$$

Resolving vertically,

$$W - N \cos \theta = 0.$$

Solving these equations for the unknown quantities N and θ, we have

$$\tan \theta = P/W;$$

$$N = P/\sin \theta = \sqrt{P^2 + W^2}.$$

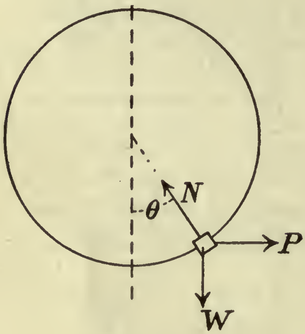


FIG. 32.

Geometrical solution.—The particle is in equilibrium under the action of three forces, P, W and N. Two of these (P and W) are known; hence the force triangle can be drawn completely, thus determining the magnitude and direction of N. From the triangle it is evident that N acts toward the center; that its magnitude is $\sqrt{P^2 + W^2}$; and that $\tan \theta = P/W$.

II. A particle whose mass is W lbs. rests upon a smooth horizontal plane; to it are attached a string AB passing over a smooth peg at B and sustaining a body of mass P lbs., and a string AC passing over a smooth peg at C and sustaining a body of mass Q lbs. Required the direction of AC for equilibrium.

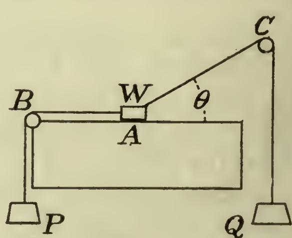


FIG. 33.

Algebraic solution.—The particle is acted upon by four forces: its weight, W lbs.; a force of P lbs. due to the tension in the string AB; a force of Q lbs. due to the tension in the string AC; and the normal pressure N due to the plane.

Let θ denote the angle between AC and the horizontal. Resolving forces horizontally and vertically,

$$-P + Q \cos \theta = 0;$$

$$N - W + Q \sin \theta = 0.$$

Solving for N and θ,

$$\cos\theta = P/Q;$$

$$N = W - \sqrt{Q^2 - P^2}.$$

The *geometrical solution* by the force triangle is obvious.

III. To determine the position of equilibrium of a body resting on the surface of a smooth elliptic cylinder and acted upon by a force of known magnitude directed along the tangent to the ellipse.

Let the mass of the body be W lbs., and let the tangential force be due to the tension in a string which passes over a smooth peg and sustains a body of known mass P lbs., as in Fig. 34.

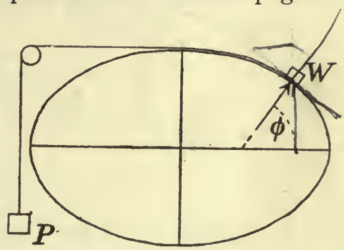


FIG. 34.

The body is acted upon by three forces: its weight W lbs. acting vertically downward; the pull of the cord, its magnitude being P lbs. and its direction that of the tangent to the ellipse; and the normal pressure exerted by the smooth surface, its magnitude N being unknown.

If ϕ denotes the angle between the normal and the axis of x, measured as shown in the figure, the equations obtained by resolving along the tangent and normal are

$$W\cos\phi - P = 0; \quad \quad (1)$$

$$N - W\sin\phi = 0. \quad \quad (2)$$

From (1), $$\cos\phi = P/W.$$

From (2), $$N = W\sin\phi = \sqrt{W^2 - P^2}. \quad \quad (3)$$

The coördinates of the point for which $\cos\phi = P/W$ may be found as follows:

$$\cos\phi = -\frac{dy}{ds} = -\frac{dy/dx}{\sqrt{1 + (dy/dx)^2}}.$$

The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \quad (4)$$

Differentiating, $$\frac{dy}{dx} = -\frac{b^2x}{a^2y};$$

which substituted in the value of cos ϕ gives

$$\cos \phi = \frac{b^2x}{\sqrt{a^4y^2 + b^4x^2}} = \frac{P}{W}. \qquad (5)$$

The coördinates of the position of equilibrium must satisfy equations (4) and (5). Solving,

$$\frac{x}{a} = \sqrt{\frac{P^2a^2}{P^2a^2 + (W^2 - P^2)b^2}};$$

$$\frac{y}{b} = \sqrt{\frac{(W^2 - P^2)b^2}{P^2a^2 + (W^2 - P^2)b^2}}.$$

Examples.

1. A body of W lbs. mass, resting on a smooth plane surface inclined at angle θ to the horizontal, is supported by the pressure of the plane and another force acting at angle α with the plane. Determine the supporting forces.

Ans. Normal pressure $= W \cos(\theta + \alpha)/\cos \alpha$.

2. In what direction must a force of P lbs. be applied to a body whose mass is W lbs. to hold it at rest on a smooth plane inclined at angle α to the horizontal?

Ans. If $\theta =$ angle between P and the plane, $\cos \theta = (W/P) \sin \alpha$.

3. A certain body may be supported on a smooth inclined plane by a force P acting at an angle α with the horizontal, or by a force Q acting at an angle β with the horizontal. Required the mass W of the body and the inclination θ of the plane to the horizontal.

Ans. $\tan \theta = \dfrac{P \cos \alpha - Q \cos \beta}{Q \sin \beta - P \sin \alpha}$. $W = \dfrac{PQ \sin (\beta - \alpha)}{P \cos \alpha - Q \cos \beta}$.

4. A certain body may be supported on a smooth inclined plane by a horizontal force of 10 lbs., or by a force of 8 lbs. acting along the plane. Required the mass of the body and the inclination of the plane. *Ans.* 13⅓ lbs.; 36° 52′.

5. A body whose mass is 10 lbs. is supported on a smooth inclined plane by a force of 2 lbs. acting along the plane and a horizontal force of 5 lbs. Determine the inclination of the plane.

6. A smooth wire, bent into the form of a parabola with axis vertical and vertex downward, carries a bead of mass W lbs. which is

pulled horizontally by a force of P lbs. Determine the position of equilibrium.

Ans. If the equation of the parabola is $x^2 = 4my$, the coördinates of the position of equilibrium are $x = (2P/W)m$, $y = (P^2/W^2)m$.

7. A heavy bead is placed upon a smooth circular wire in a vertical plane, and is pulled by a string which passes over a smooth pulley at the highest point of the circle and carries a body of known weight. Find the position of equilibrium.

8. Two particles whose masses are P lbs. and Q lbs. are connected by a flexible string of length l which passes over a smooth circular cylinder of diameter d with axis horizontal. Required the position of equilibrium. (Fig. 35.)

[Apply the conditions of equilibrium to each particle separately, remembering that the tension in the string is uniform throughout its length, so that it exerts equal forces on the two particles. Both particles, or only one of them, may be in contact with the cylinder in the position of equilibrium; these two cases must be treated separately.]

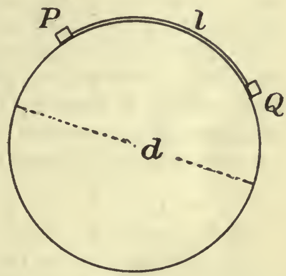


FIG. 35.

9. In the case described in Ex. 8, let the masses of the particles be 2 lbs. and 3 lbs., the length of the string 1 ft., and the diameter of the cylinder 18 ins. Determine the position of equilibrium, the tension in the string, and the reactions exerted on the particles by the cylinder.

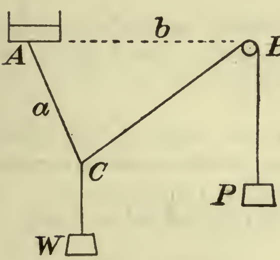


FIG. 36.

10. A string attached to a fixed point A and passing over a smooth peg B, carries at the free end a body of known weight P, and a body of weight W is suspended at a point C by means of another string. The length AC and the positions A and B being known, it is required to determine the position of equilibrium. (Fig. 36.)

[The solution involves an equation of the third degree.]

11. In Ex. 10, let the weight W be suspended from a smooth ring sliding on the string ACB. Determine the position of equilibrium.

12. Given three concurrent forces of magnitudes P, Q, R, the angles between P and Q, Q and R, R and P, respectively, being r, p, q. Prove that the square of the resultant is equal to

$$P^2 + Q^2 + R^2 + 2PQ \cos r + 2QR \cos p + 2RP \cos q.$$

13. AB and AC (Fig. 37) are smooth rods lying in a vertical plane. Heavy rings of known masses P lbs. and Q lbs. slide upon the rods, being connected by a flexible string MN. Required the inclination of MN to the horizontal in the position of equilibrium.

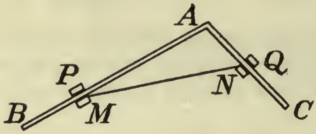


Fig. 37.

14. Two particles whose masses are P lbs. and Q lbs. respectively rest against smooth inclined planes whose inclinations to the horizontal are α and β. The particles are connected by a flexible cord which passes over a smooth peg placed vertically above the intersection of the planes. If l is the length of the cord and h the vertical distance of the peg above the intersection of the planes, prove that the angles θ and ϕ made by the cord with the planes in the position of equilibrium are determined by the equations

$$P\frac{\sin \alpha}{\cos \theta} = Q\frac{\sin \beta}{\cos \phi}; \quad \frac{\cos \alpha}{\sin \theta} + \frac{\cos \beta}{\sin \phi} = \frac{l}{h}.$$

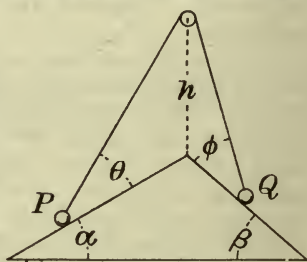


Fig. 38.

15. Determine the magnitude and direction of the least force that will sustain a particle of mass W kilograms on a smooth plane inclined at an angle α with the horizon. Compute results if $W = 45$ and $\alpha = 36°$.

16. A particle on a smooth plane inclined at angle α to the horizon is acted on by a force directed toward a fixed point and varying inversely as the square of the distance from that point. Determine the position of equilibrium.

17. Two smooth rings of 14 lbs. and 18 lbs., connected by a flexible string 2 ft. long, slide on a smooth vertical circular wire of radius 5 ft. Determine the position of equilibrium.

Ans. Angle between string and horizontal $= 1° \, 28'$.

CHAPTER IV.

COMPOSITION AND RESOLUTION OF NON-CONCURRENT FORCES IN THE SAME PLANE.

§ 1. *Two Non-Concurrent Forces.*

81. Rigid Body.— In the discussions which follow, the systems of forces are in most cases regarded as applied to a rigid body.

A *rigid body* may be defined as one whose particles do not change their positions relative to one another under any applied forces. No known body satisfies this condition strictly, even for forces of small magnitude, but in the solution of problems of practical importance most solid bodies may, with slight error, be regarded as rigid. After a body has assumed a form of equilibrium under applied forces it may, in applying the principles of Statics, be treated as a rigid body without error.

82. Change of Point of Application.— The effect of a force upon the motion of a rigid body will be the same, at whatever point in its line of action it is applied, if the particle upon which it acts is rigidly connected with the body.

This proposition, which is amply justified by experience, is fundamental to the development of the principles of Statics. In applying the principle, we are at liberty to assume a point of application outside the actual body, the latter being ideally extended to any desired limits.

83. Collinear Forces.— Two forces having the same line of action, applied to the same rigid body at any points of that line, may be combined as if they were concurrent; since by Art. 82 both may be treated as if applied at any one point in their common line of action.

As a particular case, two forces applied to a rigid body balance each other if they are equal in magnitude, opposite in direction, and collinear. Hence such a pair of forces may be introduced into a system without changing its effect.

84. Resultant of Two Non-Parallel Forces.— If two coplanar forces are not parallel, their lines of action must intersect, and the

4

point of intersection may be treated as their common point of application. They may therefore be treated as concurrent forces, and their resultant may be determined as in Art. 59. Hence the following proposition may be stated:

The resultant of two non-parallel forces acting in the same plane on a rigid body is a force equal to their vector sum, and its line of action passes through the point of intersection of the lines of action of the given forces. Its point of application may be any point of this line.

Thus, if two forces are applied to a body at points M and N (Fig. 39), their lines of action being MS and NT, each may be assumed to act at R, the point of intersection of these lines. The resultant must therefore be a force which may be taken as acting at R; that is, its line of action must pass through R. When the magnitude and direction of the resultant have been found by constructing the vector triangle, the line of action becomes known, and any point of this line may be regarded as the point of application.

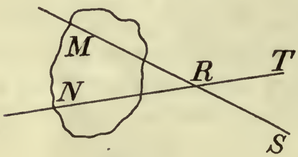


FIG. 39.

By an extension of the above reasoning, any number of forces *whose lines of action meet in a point* may be treated as if that point were their common point of application; in other words, as if they were concurrent. This is true even though the point of intersection of the lines of action falls outside the limits of the body. Thus, if the forces P_1, P_2, P_3 are applied to the bar AC (Fig. 40) at points A, B and C, their directions being such that their lines of action intersect at a point D, their effect is the same as if all were applied at D, the latter point being regarded as rigidly connected with the bar.

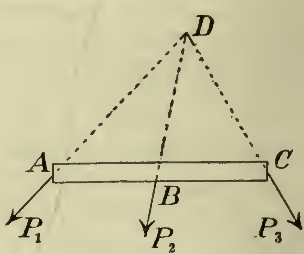


FIG. 40.

85. Resolution of a Force Into Two Non-Parallel Components.— Let M (Fig. 41) be the point of application of a force, and MS its line of action. Any point in this line, as R, may be

regarded as the point of application, and the force may be replaced by two components acting at R. The magnitudes and directions of the two components must be such that their vector sum is equal to the given force.

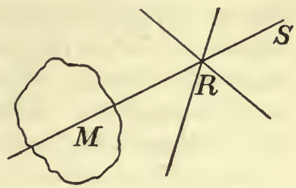


FIG. 41.

86. Resultant of Two Parallel Forces.— In determining the resultant of two parallel forces, two cases must be considered, according as the forces act in the same direction or in opposite directions.

(1) *Forces having the same direction.*— Let P and Q denote the magnitudes of two forces having the same direction, A and B (Fig. 42) being any points in their lines of action. Let two forces, each of magnitude F, be applied

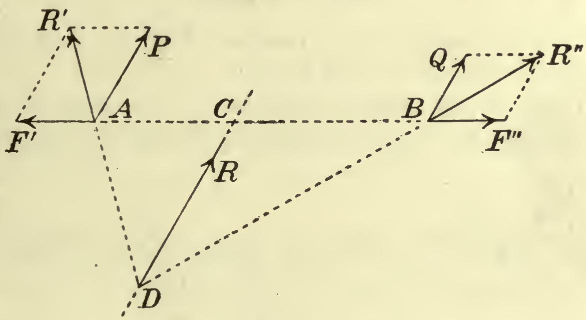


FIG. 42.

to the body, acting in opposite directions along the line AB; call these forces F' and F''. The system of four forces (P, Q, F', F'') is equivalent to the given system of two forces (P, Q), since F' and F'' balance each other. Let the force F' be combined with P (giving a resultant R') and the force F'' with Q (giving a resultant R''). The line of action of R' passes through A, and its direction is that of the diagonal of a parallelogram with sides drawn from A, proportional and parallel to P and F. The line of action of R'' passes through B, and is found by constructing a parallelogram with sides drawn from B, proportional and parallel to Q and F. The forces R' and R'' may be regarded as applied at D, the intersection of their lines of action; R' is therefore

equivalent to two forces equal and parallel to P and F' applied at D, and R'' is equivalent to two forces equal and parallel to Q and F'' applied at D; these four forces applied at D being thus equivalent to the given forces P and Q. But since the forces F' and F'' balance each other, the system is equivalent to two collinear forces of magnitudes P and Q, or to a single force $P+Q$, acting at D parallel to the given forces. This single force is therefore the resultant of the two given forces.

It remains to determine the position of the line of action of this resultant relative to the lines of action of the given forces. Let C be the point in which this line intersects AB; then by similar triangles,

$$F/P = AC/CD, \text{ and } F/Q = CB/CD.$$

Combining these two equations,

$$AC/CB = Q/P;$$

that is, the line of action of the resultant divides AB into segments inversely proportional to P and Q.

(2) *Forces having opposite directions.*— The case in which P and Q have opposite directions is represented in Fig. 43; the reasoning

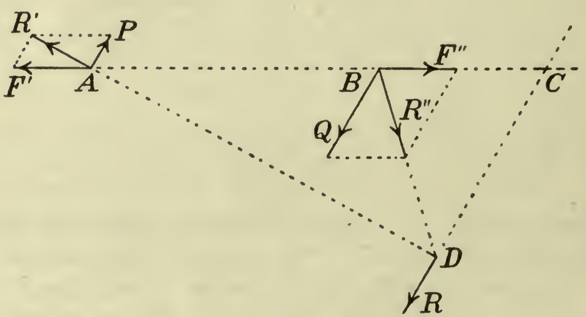


FIG. 43.

employed in the preceding case applies equally to this, the two figures being lettered in a corresponding manner. The magnitude of the resultant is found to be the difference (or algebraic sum) of P and Q. Also, the point D and therefore the line CD will fall outside the space included between the lines of action of P and Q and on the side of the greater of these forces. In this case the point C divides AB *externally* into segments inversely proportional to P and Q.

The examples which follow are designed to illustrate the method of finding the resultant of two parallel forces, or of finding two parallel forces that are equivalent to a single force. The reader who is already acquainted with the principles of equilibrium (to be developed later) will notice that several of the examples can be solved by means of these principles. It is, however, desirable that they should be analyzed by the principles of composition and resolution of forces already developed; the resultant of two forces being regarded as a single force which may replace them without affecting the state of the body as regards rest or motion.

Thus, in example 3 of the following list, the bar AB may be acted upon by any number of forces not specified; these, together with the two upward forces exerted by the strings, maintain the bar in its condition of rest. Now suppose the action of these two forces ceases; what single force may be applied with the same result?

Examples.

1. Parallel forces of 40 lbs. and 65 lbs. acting in the same direction are applied at the ends of a bar 12 ft. long. Find the magnitude, direction and line of action of a single force which would produce the same effect.

2. Assuming the two forces in the preceding example to act in opposite directions, determine their resultant completely.

3. A heavy bar AB (Fig. 44), 8 ft. long, is supported by vertical cords attached at A and B, which pass over smooth pegs at C and D and sustain bodies P and Q of masses 100 lbs. and 60 lbs. respectively. If the two supporting cords are replaced by a single cord, at what point must it be attached, and what force must it exert?

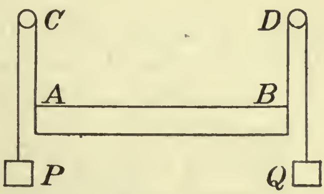


FIG. 44.

4. A heavy bar AB, 12 ft. long, is found to balance on a smooth support at C, 7.5 ft. from one end, the upward pressure exerted by the support being 48 lbs. If the same bar rests on two smooth supports at the ends, what will be the supporting forces?

5. A heavy bar 18 ft. long is supported at the ends by equal vertical forces of 40 lbs. If the supports are moved to points 4 ft. from one end and 6 ft. from the other respectively, what are the supporting forces?

6. The bar AB (Fig. 45) carries a load P of 20 lbs. hanging by a string from the end A, and is supported by a string attached at C, 4 ft. from A, the tension of the supporting string being found by a spring balance to be 40 lbs. If the load P is removed, at what point must the string be attached in order to support the bar? What tension will it sustain?

Ans. 8 ft. from A; 20 lbs.

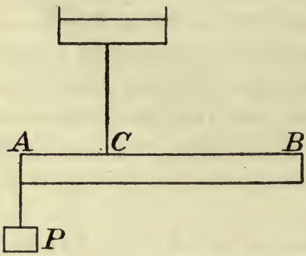


Fig. 45.

7. It is found that a bar AB, 12 ft. long, may be supported by a force of 18 lbs. applied at the middle point. If it rests in a horizontal position against smooth supports at A and C, 4 ft. apart, the support at A being above and that at C below the bar, what will be the supporting forces?

Ans. 27 lbs. upward at C, 9 lbs. downward at A.

87. Two Equal and Opposite Forces.— If two forces P and Q are equal in magnitude, opposite in direction, and have different lines of action, the construction above given (Art. 86) for finding their resultant fails, for the reason that the two lines AD and BD (Fig. 43) become parallel. If it is attempted to apply the general rule for determining the resultant of two opposite forces, it is found that the magnitude of the resultant $P-Q$ is zero; while the equation for determining its line of action becomes (see Fig. 43)

$$AC/CB = Q/P = 1,$$

which can be true only if AC and CB are infinite. Mathematically interpreted, these results mean that the resultant is a force of magnitude zero, acting in a line infinitely distant from the lines of action of the given forces.

Evidently P and Q in this case form a couple (Art. 53). A further discussion of couples will be given later.

88. Moment of Resultant of Two Forces. — *Proposition.* — The algebraic sum of the moments of any two coplanar forces with respect to a point in their plane is equal to the moment of their resultant with respect to that point.

In the proof of this proposition two cases must be considered, according as the two given forces are or are not parallel.

(*a*) *Non-parallel forces.*— For two non-parallel forces, the prop-

osition just stated is a special case of that proved in Art. 29 for any localized vectors. For the resultant of two non-parallel forces is a force equal to their vector sum, acting in a line through the intersection of their lines of action.*

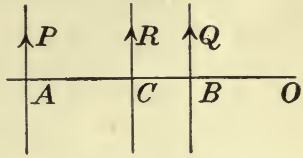


FIG. 46.

(*b*) *Parallel forces.*—Let P and Q represent the magnitudes of any two parallel forces and R that of their resultant. Take any point O (Fig. 46) as origin of moments, and draw through O a line perpendicular to the forces, intersecting their lines of action at A, B and C respectively. Let $OA = p$, $OB = q$, $OC = r$. From Art. 86 we have

$$\frac{P}{Q} = \frac{BC}{AC} = \frac{r-q}{p-r}.$$

Reducing,

$$Pp + Qq = (P + Q)r = Rr.$$

If the forces P and Q have opposite directions, the demonstration needs modification. (See Fig. 47.) The equation is

$$\frac{P}{Q} = \frac{BC}{AC} = \frac{r-q}{r-p};$$

$$\therefore \qquad Pp - Qq = (P - Q)r = Rr.$$

In both cases, the reasoning is easily adapted to the case in which the origin of moments falls between the lines of action of the two forces P and Q. In all cases, the result is expressed by the proposition that the moment of the resultant of two parallel forces is equal to the algebraic sum of their moments.

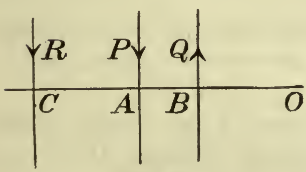


FIG. 47.

The proof of the proposition for parallel forces may be put in another form, as follows: Referring to Art. 86, let any point (Figs. 42 and 43) be taken as origin of moments. Having proved that the moment of the resultant of two non-parallel

* Any two non-parallel forces may, in fact, as already shown, be treated as if concurrent, so that this case reduces to that treated in Art. 71.

forces is equal to the algebraic sum of their separate moments, we may write the following equations:

$$\text{mom. of } R' = \text{mom. of } P + \text{mom. of } F';$$

$$\text{mom. of } R'' = \text{mom. of } Q + \text{mom. of } F'';$$

$$\begin{aligned}\text{mom. of } R &= \text{mom of } R' + \text{mom. of } R'' \\ &= \text{mom. of } P + \text{mom. of } Q + \\ &\qquad \text{mom. of } F' + \text{mom. of } F'' \\ &= \text{mom. of } P + \text{mom. of } Q\end{aligned}$$

(since the moments of F' and F'' are equal in magnitude but opposite in sign).

89. Moment of a Couple.—The proposition just proved is meaningless when applied to the case of two equal and opposite forces, since two such forces are not equivalent to a single resultant force. The moment of a couple will therefore be made a matter of definition, as follows:

The moment of a couple is the algebraic sum of the moments of its two forces.

With this definition it may readily be shown that *the moment of a couple has the same value for every origin in its plane, and is equal to the product of the magnitude of either force into the perpendicular distance between the lines of action.*

§ 2. *Couples.*

90. Equivalent Couples in the Same Plane.— Any two couples in the same plane are equivalent if their moments are equal in magnitude and sign.

To prove this, consider first the resultant of two couples whose moments are equal in magnitude but opposite in sign. Let P denote the magnitude of the forces of one couple and p the length of the arm or perpendicular distance between the lines of action; and let Q, q denote the like quantities for the second couple, these quantities being so related that

$$Pp = Qq.$$

Let the lines of action and directions of the four forces be as shown

in Fig. 48, so that the moments have opposite signs. It may be shown that their resultant is zero.

Notice first that the lengths of the sides of the parallelogram

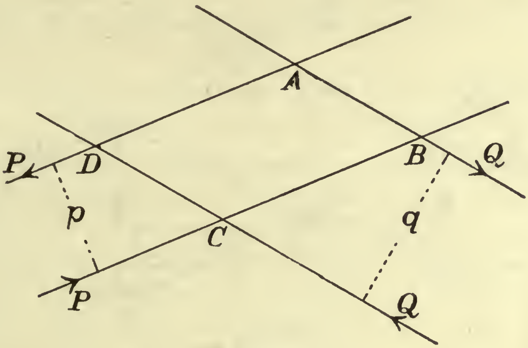


FIG. 48.

$ABCD$ formed by the four lines of action are proportional to P and Q. For

$$\text{area } ABCD = (AB) \cdot q = (AD) \cdot p;$$

and since $Pp = Qq$,

$$AB/AD = Q/P.$$

The resultant of the force P acting along AD and the force Q acting along AB must therefore act along the diagonal AC of the parallelogram $ABCD$; and the resultant of the force P acting along CB and the force Q acting along CD must also act along the diagonal CA. But these resultants are equal, opposite, and collinear; hence their resultant is zero. The resultant of the two couples is therefore zero.

Next, starting with the given couple (P, p), let two pairs of equal and opposite forces of magnitude Q be assumed to act in the lines AB, CD (Fig. 49), so taken that $Qq = Pp$. These four forces form two couples; one of these counterbalances the given couple as shown above, hence the other must be equivalent to the given couple. That is, the given couple (P, p) is equivalent to the couple (Q, q) whose moment is equal to its own in magnitude and sign.

Since the lines AB and DC may be any two parallel lines intersecting the lines of action of the given couple, it follows that any two

coplanar couples whose forces are not parallel are equivalent if their moments are algebraically equal.

But two couples whose forces are parallel and moments equal are

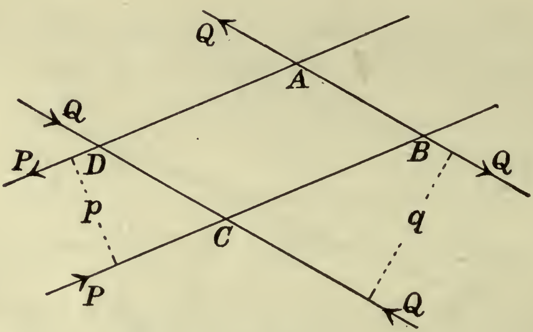


FIG. 49.

equivalent, because each may be proved equivalent to a third couple; so that the proposition holds for any coplanar couples.

91. Couples in Parallel Planes.— Couples in parallel planes are equivalent if their moments are equal in magnitude and sign.

Let AB (Fig. 50) be a line perpendicular to the forces of a couple, intersecting their lines of action in A and B. Let CD be any line equal and parallel to AB, not necessarily in the plane of the couple. Introduce at C two equal and opposite forces, each equal and parallel to P, the force of the given couple; also two similar forces at D. The force at A and the upward force at D may be replaced by an upward force $2P$ acting at the middle point of AD. The force at B and the downward force at C may be replaced by a downward force $2P$ acting at the middle point of CB. These two forces $2P$ balance each other, leaving two forces,— an upward force at C and a downward force at D, each equal to P. These form a couple equivalent to the given couple.

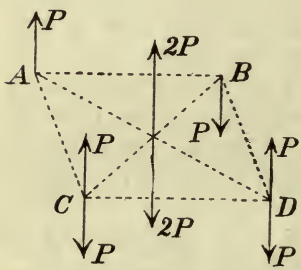


FIG. 50.

Since the couple acting at C and D is equivalent to any couple in the same plane having an equal moment; and since CD might be

taken in any plane parallel to that of the given couple, it follows that any two couples in parallel planes are equivalent if their moments are equal.

92. **Equivalence of Couples—General Result.**—The results of the last two Articles may be stated in one general proposition, as follows:

Any two couples in the same plane or in parallel planes are equivalent if their moments are equal in magnitude and sign.

93. **Resultant of Coplanar Couples.**—The resultant of any number of coplanar couples is a couple whose moment is equal to the algebraic sum of the moments of the given couples.

Choose any two parallel lines in the plane of the couples, as MN, $M'N'$ (Fig. 51), and let each of the given couples be replaced by an equivalent couple with forces acting in these lines. Let the common arm of these substituted couples be p, and let their forces be P_1, P_2, P_3, etc., their values being either positive or negative according to the directions of the forces, and being such that $P_1 p$, $P_2 p$, . . . are algebraically equal to the moments of the given couples. The resultant of the forces acting in MN is a force equal to their algebraic sum $P_1 + P_2 + P_3 +$ . . . , while the resultant of the forces acting in $M'N'$ has the same magnitude but the opposite direction. Hence the whole system is equivalent to a couple of moment

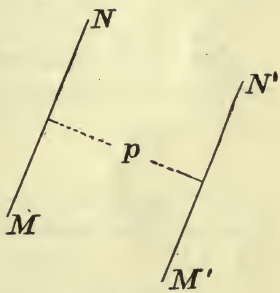


FIG. 51.

$$(P_1 + P_2 + P_3 + \quad . \quad . \quad . \quad) p,$$

which is equal to

$$P_1 p + P_2 p + P_3 p + \quad . \quad . \quad . \quad ,$$

that is, to the algebraic sum of the moments of the given couples.

The proposition obviously holds for couples lying in any parallel planes.

94. **Resultant of Force and Couple.**—A force and a couple acting in the same plane or in parallel planes are equivalent to a single force.

Let P be the magnitude of the given force and G the moment of

the given couple. Replace the couple by an equivalent couple in a plane parallel to its own, containing the given force P. The magnitude, direction and line of action of one of the forces of this equivalent couple may be chosen arbitrarily; let them be so chosen that this force counterbalances the given force P. The other force of the couple must be equal in magnitude and direction to P, and its line of action must be such that the moment of the couple is algebraically equal to G; its line of action is therefore at a distance G/P from that of the given force. This last force is the resultant of the given force and couple.

Examples.

1. A force of 50 lbs. acting in any assumed line, and a couple coplanar with it whose arm is 29 ft. and whose forces have the magnitude 10 lbs., have what resultant?

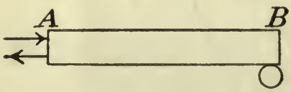


Fig. 52.

2. A bar AB, resting horizontally upon a smooth support at B, is held in equilibrium by two opposite horizontal forces applied at the end A in lines 6 ins. apart. The supporting force at B is 5 lbs. and the horizontal forces are each 50 lbs. If the horizontal forces cease to act and the support is removed, what single force will support the bar in equilibrium? (Fig. 52.)

§ 3. *Any Number of Coplanar Forces.*

95. Resultant of Any Number of Forces.— It may now be shown that the resultant of any number of coplanar forces is a single force equal to their vector sum, unless this sum is zero; in which case the resultant is a couple.

Of three or more forces, there must be two which do not form a couple. Any such pair may be replaced by their resultant, which is a single force equal to their vector sum (Arts. 84, 86), thus reducing the number of forces by one. This process may be repeated until the system has been reduced to an equivalent system of two forces. If these two are not equal and opposite, they are equivalent to a single force equal to their vector sum and therefore to the vector sum of the given forces. If they are equal and opposite, they form a couple which is the resultant of the given system; the vector sum of all the forces being zero.

96. Moment of Resultant of Any Number of Forces.— In a similar manner the principle of moments, already proved for two forces, may be extended to any number of coplanar forces. That is, it may be shown that *the sum of the moments of any number of coplanar forces about any origin in their plane is equal to the moment of their resultant about that origin.*

Combining two of the forces which are not equal and opposite, the sum of their moments is (by Art. 88) equal to the moment of their resultant. Combining this resultant with another force not equal and opposite to it, the sum of their moments is equal to the moment of *their* resultant. This process may be continued until the number of forces is reduced to two. If these two are not equal and opposite, the sum of their moments (equal to the sum of the moments of the given forces) is equal to the moment of the single force which is their resultant and the resultant of the given system. If the two forces are equal and opposite, they form a couple which is the resultant of the given system, and whose moment is equal to the sum of the moments of the given forces. Hence the proposition is proved.

97. Computation of Resultant.— From the foregoing principles, the resultant of any system of coplanar forces may be computed, whether this resultant be a single force or a couple.

(1) If the resultant is a single force, its magnitude and direction may be found as if the forces were concurrent (Art. 65 or 69); the position of its line of action must be such that its moment about any assumed point is equal to the sum of the moments of the given forces about that point. Thus, if P denotes the magnitude of the resultant force, and G the sum of the moments of the given forces, the line of action of the resultant must be at a distance G/P from the origin of moments. This condition, together with the requirement that the sign of the moment of the resultant is the same as that of the sum of the moments of the given forces, completely determines the line of action of the resultant.

(2) If the vector sum of the given forces is zero, the resultant of the corresponding concurrent system will be zero. The resultant of the given system is then a couple, whose moment may be found by computing the sum of the moments of the given forces about any point. The value of this moment will be the same, whatever point be chosen as origin.

Let the following examples be solved by the direct application of these principles.

EXAMPLES.

[Let the magnitude, direction and point of application of a force be specified by the following notation: x, y denote the rectangular coördinates of its point of application; α the angle its direction makes with the positive direction of the x-axis; P its magnitude.]

1. Find the resultant of the following forces:

$P_1 = 20$ lbs., $x_1 = 2$ ft., $y_1 = 6$ ft., $\alpha_1 = 0°$;
$P_2 = 50$ " $x_2 = 3$ " $y_2 = 7$ " $\alpha_2 = 180°$;
$P_3 = 80$ " $x_3 = -5$ " $y_3 = 7$ " $\alpha_3 = 90°$.

2. Find the resultant of a system of parallel forces whose magnitudes and directions are 10 lbs., 24 lbs., —15 lbs., 3 lbs., —48 lbs.; the successive distances between their lines of action being 2 ft., 3 ft., 8 ft., 7 ft.

3. Find the resultant of the following forces:

$P_1 = 40$ lbs., $x_1 = 4$ ft., $y_1 = 6$ ft., $\alpha_1 = 0°$;
$P_2 = 27$ " $x_2 = 2$ " $y_2 = 14$ " $\alpha_2 = 180°$;
$P_3 = 13$ " $x_3 = -6$ " $y_3 = -8$ " $\alpha_3 = 180°$;
$P_4 = 16$ " $x_4 = 10$ " $y_4 = 0$ " $\alpha_4 = 90°$;
$P_5 = 20$ " $x_5 = -6$ " $y_5 = 2$ " $\alpha_5 = 90°$;
$P_6 = 36$ " $x_6 = -12$ " $y_6 = 3$ " $\alpha_6 = 270°$.

98. Reduction to Force and Couple.— The general results above found regarding the resultant of any coplanar forces may be deduced by a somewhat different line of reasoning, as follows:

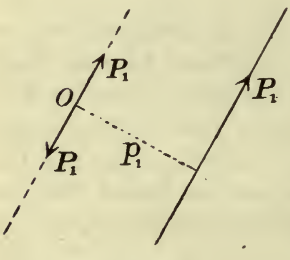


FIG. 53.

Let P_1, P_2, . . . , represent the magnitudes of any coplanar forces applied to a rigid body. Choose any point O in the plane of the forces (Fig. 53), and let the perpendicular distances from O to the lines of action of the several forces be p_1, p_2, etc. Consider first some one force as P_1. Suppose two equal and opposite forces, parallel to P_1 and equal to it in magnitude, to be applied to the body at O. One of these forms, with the given force, a couple of moment $P_1 p_1$. Hence the given force P_1 is equivalent to a couple of moment $P_1 p_1$ and a force equal and parallel to P_1 applied at O. If this process is repeated for each of the given

forces, it is seen that the given system is equivalent to the following two systems:

(1) A system of concurrent forces, applied at O, equal and parallel to $P_1, P_2, \quad . \quad . \quad . \quad .$

(2) A system of couples whose moments are $P_1p_1, P_2p_2, \quad . \quad . \quad . \quad .$

The resultant of this system of concurrent forces may be found as in Art. 69, while the couples may be combined as in Art. 93. Hence the following proposition:

Any system of coplanar forces is equivalent to a single force equal to their vector sum, applied at any chosen point; together with a couple whose moment is the algebraic sum of the moments of the given forces with respect to that point.

99. **Computation of the Force and Couple.**— Let the point O at which the concurrent forces are assumed to act be taken as origin of coördinates, and let the angles made with the x- and y-axes by any force P be denoted by α, β, with proper suffix.

Let the force and couple to which the system is reduced be specified by the following notation:

$$R = \text{magnitude of force;}$$
$$a = \text{angle between } R \text{ and } x\text{-axis;}$$
$$b = \text{angle between } R \text{ and } y\text{-axis;}$$
$$X, Y = \text{axial components of } R\text{;}$$
$$G = \text{moment of couple.}$$

Then

$$\left.\begin{aligned} X &= R\cos a = P_1\cos\alpha_1 + P_2\cos\alpha_2 + \quad . \quad . \quad . \quad ; \\ Y &= R\cos b = P_1\cos\beta_1 + P_2\cos\beta_2 + \quad . \quad . \quad . \quad ; \end{aligned}\right\} \quad (1)$$

$$R = \sqrt{X^2 + Y^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\cos a = X/R; \ \cos b = Y/R. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also, by Art. 98,

$$G = P_1p_1 + P_2p_2 + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The values of R, a and b completely determine the required force; and the value of G determines the moment of the couple.

It is seen that the magnitude and direction of the force R are the same wherever the point O be taken; while in general the value of G depends upon the position of the point at which the concurrent forces are assumed to act.

Let the following examples be solved by this process.

EXAMPLES.

1. Find a force and couple equivalent to the following forces:

$P_1 = 70$ lbs., $x_1 = 4$ ft., $y_1 = 8$ ft., $\alpha_1 = 60°$;
$P_2 = 40$ " $x_2 = 6$ " $y_2 = 6$ " $\alpha_2 = 45°$;
$P_3 = 25$ " $x_3 = 0$ " $y_3 = -10$ " $\alpha_3 = 120°$.

2. Reduce each of the systems of forces given in examples 1 and 3, Art. 97, to a couple and a force applied at the origin of coördinates.

100. Resultant Force or Resultant Couple.— The force and couple to which any system may be reduced by the above process may in general be combined into a simpler resultant. Since in particular cases one or both the quantities R and G determined by the above process may reduce to zero, the following four cases must be considered:

(1) Suppose neither R nor G is equal to zero. In this case the force and couple may be combined into a single force (Art. 94) having the same magnitude and direction as R; its line of action being distant G/R from O.

(2) Suppose $G = 0$. In this case the whole system reduces to the single force R, its line of action passing through the assumed point O. [This case always results if the given system is equivalent to a single resultant force, and if the point O happens to be chosen upon its line of action.]

(3) Let $R = 0$. In this case the resultant of the given system is a couple whose moment is G.

(4) Let $R = 0$, $G = 0$. In this case the given system is equivalent to no force. In other words the given forces exactly balance each other and the system is in equilibrium.*

EXAMPLES.

In each of the examples of Art. 99, determine completely the resultant force or resultant couple.

---

* In the last three Articles we have reached by means of algebraic analysis the same results deduced previously by geometrical reasoning. The student will find it profitable to become familiar with both methods of treatment.

CHAPTER V.

EQUILIBRIUM OF COPLANAR FORCES.

§ 1. *General Principles of Equilibrium.*

101. **Meaning of Equilibrium.**— The word equilibrium is used with reference both to *forces* and to *bodies*. (See Art. 57.)

Any number of forces form a *system in equilibrium* if their combined action produces no effect upon the motion of the body to which they are applied.

A *rigid body* is in equilibrium if all external forces acting upon it form a system in equilibrium.

102. **General Condition of Equilibrium.**— It follows from the definition of equilibrium that the *necessary and sufficient condition* of equilibrium for any system of forces is that their resultant is zero. This general condition implies subordinate conditions which are now to be considered.

These subordinate conditions may be deduced in two ways: First, by the direct application of the principles regarding the resultant deduced in Arts. 95 and 96; or second, by applying the results of the algebraic discussion of Arts. 98–100. It will be useful to consider both methods.

103. **Equations of Equilibrium.**— It has been shown (Art. 95) that the resultant is either a force or a couple. If a force, it is equal to the vector sum of the given forces, and the resolved part of the resultant in any chosen direction must equal the algebraic sum of the resolved parts of the given forces in that direction. If the resultant is a couple, the vector sum of the forces is zero, and the algebraic sum of their resolved parts in any direction is zero. In either case, the sum of the moments of the given forces about any origin is equal to the moment of the resultant (Art. 96). Hence, if the resultant is zero, the following conditions must be satisfied:

(1) The sum of the resolved parts of the given forces in any direction is zero.

(2) The sum of the moments of the given forces about any point is zero.

From these two principles, an infinite number of equations may be written; since the forces may be resolved in any direction, and any point may be taken as origin of moments. It may be shown, however, that only three of these equations can be independent.

104. **Three Independent Equations.** — In order to determine how many of the possible equations of equilibrium can be independent, consider how much is implied by any one of them.

(*a*) If the sum of the resolved parts of the forces is zero for any given direction of resolution, the resultant force, if one exists, must be perpendicular to that direction. There may, however, be a resultant couple.

(*b*) If the sum of the moments is zero for any chosen origin, there can be no resultant couple, since the moment of a couple is not zero for any origin; if there is a resultant force, its line of action must pass through the origin.

These principles lead immediately to the following propositions:

(1) There will be equilibrium if the sum of the resolved parts is zero for each of two directions of resolution, and the sum of the moments is zero for one origin.

For, by principle (*a*), there can be no resultant force, since such a force would need to be perpendicular to both directions of resolution; and by principle (*b*) there can be no resultant couple, since the moment of a couple cannot be zero.

(2) There will be equilibrium if the sum of the moments is zero for each of the two origins, and the sum of the resolved parts is zero in any one direction, not perpendicular to the line joining the two origins.

For, by principle (*b*), there can be no resultant couple, and the resultant force (if one exists) must act in a line through the two origins of moments; while by principle (*a*), the resultant force (if there is one) must be perpendicular to the direction of resolution.

(3) There will be equilibrium if the sum of the moments is zero for each of three origins not in the same straight line.

For, by principle (*b*), the resultant force (if one exists) must act in a line containing the three origins; and there can be no resultant couple, since the moment of a couple cannot be zero.

Since, therefore, three equations can be written, which, if satisfied, insure that the resultant is zero, it follows that all possible equations obtained in accordance with the principles of Art. 103 must be true

if those three are true. Hence three, and only three, of the equations of equilibrium can be independent.

105. Conditions of Equilibrium Deduced Algebraically.— It was shown in Art. 98 that any system of coplanar forces can be reduced to a force and a couple; the force being applied at any chosen point and being equal to the vector sum of the given forces, while the moment of the couple is equal to the algebraic sum of the moments of the given forces about the chosen point.

Using the notation of Art. 99, the equations for computing the force and couple are

$$X = R \cos a = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \quad . \quad . \quad . \quad ;$$

$$Y = R \sin a = P_1 \cos \beta_1 + P_2 \cos \beta_2 + \quad . \quad . \quad . \quad ;$$

$$R^2 = X^2 + Y^2;$$

$$G = P_1 p_1 + P_2 p_2 + \quad . \quad . \quad . \quad .$$

The analysis of the four possible cases, given in Art. 100, shows that there will be equilibrium if R and G are both zero, but not otherwise.

Now the condition $R = 0$ requires that $X = 0$ and $Y = 0$, unless either X^2 or Y^2 is negative, that is, unless X or Y is imaginary. But if $P_1, P_2, \quad . \quad . \quad .$ are real forces, X and Y must be real.

For equilibrium, therefore, it is necessary that the three equations

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \quad . \quad . \quad . \quad = 0, \quad . \quad . \quad (1)$$

$$P_1 \cos \beta_1 + P_2 \cos \beta_2 + \quad . \quad . \quad . \quad = 0, \quad . \quad . \quad (2)$$

$$P_1 p_1 + P_2 p_2 + \quad . \quad . \quad . \quad = 0, \quad . \quad . \quad (3)$$

shall be satisfied. And conversely, if these three equations are satisfied, the system is in equilibrium.

Since the origin and axes may be chosen at pleasure, an infinite number of sets of equations similar to (1), (2) and (3) may be written. If one of these sets is satisfied, all others must be.

106. Parallel Forces.— The general conditions of equilibrium deduced above apply to any system of coplanar forces. In case all the forces are parallel, however, *only two independent equations of equilibrium* can be written. For, from principles (a) and (b), Art. 104, it is evident that a system of parallel forces will be in equilibrium in either of the following cases :

(1) If the sum of the moments is zero for each of two origins not lying on a line parallel to the forces.

(2) If the sum of the moments is zero for one origin, and the sum of the resolved parts is zero for any direction not perpendicular to the forces.*

107. Special Condition of Equilibrium.— If any system of forces in equilibrium be divided into two sets, the resultants of these sets must be equal and opposite and have the same line of action.

If three forces are in equilibrium, their lines of action must meet in a point, or be parallel. For the resultant of any two must be equal and opposite to the third, and have the same line of action.

This principle is often found useful in the solution of problems in equilibrium.

§ 2. *Application of Principles of Equilibrium.*

108. General Method of Solving Problems in Equilibrium.— The solution of problems in equilibrium is the most important of the practical applications of the principles of Statics. The problems to be solved are of the following kind:

A body is in equilibrium under the action of any number of forces, some of which are partly or wholly unknown ; it is required to determine these unknown forces completely.

The general method of solving such a problem is to write three independent equations of equilibrium, introducing as many unknown quantities as necessary to represent completely all the forces acting on the body. The unknown quantities must then be determined by solving the equations.

If the number of unknown quantities is greater than three for non-parallel and non-concurrent forces (or greater than two for parallel or concurrent† forces), the equations of equilibrium are not sufficient for their determination. The problem is then indeterminate, unless additional equations can be written from the geometrical relations.

---

* Placing the sum of the resolved parts in any direction equal to zero, the same equation results, whatever the direction of resolution; all such equations reduce to the form, "algebraic sum of forces = o."

† Forces applied to the same rigid body may be regarded as concurrent if their lines of action intersect in a single point, even if they are not actually applied at that point. (Art. 82.)

The equations of equilibrium may often be simplified by a judicious selection of the directions of resolution and of the origins of moments. Thus, an equation free from any one force may be obtained by resolving in a direction perpendicular to that force, or by taking moments about an origin lying on its line of action. If moments are taken about the point of intersection of the lines of action of two forces, neither of these forces will enter the resulting equation.

In solving problems in the equilibrium of rigid bodies, the student may be aided by the following outline of the method of procedure:

(1) Specify the body to which the conditions of equilibrium are to be applied.

(2) Enumerate all forces acting upon this body, specifying the magnitude and direction of each so far as known.

(3) Write three independent equations of equilibrium (two, if the forces are known to be concurrent or parallel), introducing as many unknown quantities as necessary. In writing these equations, notice what directions of resolution, or what origins of moments, give the simplest equations.

(4) Notice whether the number of equations is as great as the number of unknown quantities. If not,

(5) Notice whether any geometrical equations can be written. Write as many of these as possible.

(6) Notice whether the number of statical and geometrical equations together is sufficient for the determination of the unknown quantities.

(7) If the problem is found to be determinate, solve the equations algebraically, thus determining the unknown quantities.

After some experience the student will be able in most cases to select readily the best methods of writing the equations of equilibrium, and will often be able to solve problems by short methods. For example, the principle stated in Art. 107 will often be found useful. The beginner will, however, usually find it useful to analyze the problem completely in a manner similar to that outlined above.

In all cases the three equations of equilibrium should be written, even if the complete solution of the problem is found to be difficult or impossible. When these have been correctly written, the problem is solved so far as the application of the principles of Mechanics is

concerned. The remainder of the process is a matter of algebraic manipulation.

This general method will now be illustrated.

109. Problems in Equilibrium of a Rigid Body.— I. A rigid bar AB (Fig. 54) rests with the end A against a horizontal floor and a vertical wall, the other end being supported by a string BC, and a weight of 20 lbs. being suspended by a string from the point B. Let AB and BC each make an angle of 30° with the horizontal, and let the weight of the bar be 5 lbs., its point of application being taken as the middle point of AB. Determine all forces acting upon the bar.

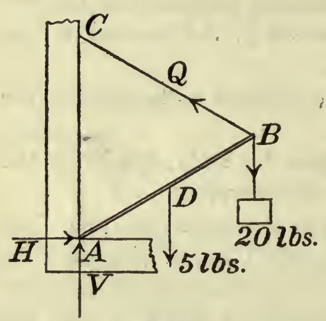


Fig. 54.

Solution.— Following the method outlined above, we proceed as follows:

(1) The bar AB is the body to which the conditions of equilibrium will be applied.

(2) The forces acting on the bar are five in number: a force of 20 lbs. vertically downward at B, due to the suspended weight; a downward force of 5 lbs. at D, the center of gravity of the bar; a force applied at B in the direction BC, due to the supporting string; a pressure at A exerted by the floor; and a pressure at A due to the wall. The last two forces may be replaced by their resultant, which is a force in some unknown direction, but applied at A. Let H and V represent the horizontal and vertical components of this force; and let Q denote the force applied at B along the line BC.

(3) We will choose as the equations of equilibrium (a) a moment equation with origin A; (b) a moment equation with origin C; and (c) a resolution equation, resolving along AC. Denoting the length of the bar by $2a$, and noticing that the triangle ABC is equilateral, the three equations are as follows:

$$Q \cdot 2a \sin 60^\circ - 20 \cdot 2a \cos 30^\circ - 5 \cdot a \cos 30^\circ = 0; \quad (a)$$

$$H \cdot 2a - 20 \cdot 2a \cos 30^\circ - 5 \cdot a \cos 30^\circ = 0; \quad (b)$$

$$V + Q \cos 60^\circ - 5 - 20 = 0. \quad (c)$$

(4, 5, 6) These equations are just sufficient to determine the three unknown quantities. No geometrical equations are needed; in fact the geometrical relations have been wholly taken account of in writing the three equations of equilibrium.

(7) From equation (a), $Q = 22.5$ lbs. From (b), $H = 19.48$ lbs. From (c), $V = 13.75$ lbs.

The resultant of the forces exerted by the floor and wall at A is equal to the resultant of H and V. Its magnitude is $\sqrt{H^2 + V^2} = 23.85$ lbs.; its direction is inclined to the horizontal at an angle θ such that $\tan \theta = V/H = 0.706$, hence $\theta = 35° \ 13'$.

The components of P which are exerted by the floor and wall respectively cannot be determined without further data. If the floor and wall are supposed perfectly smooth, the pressure exerted by each is normal to itself; hence in this case V is the force exerted by the floor and H the force exerted by the wall.

Geometrical solution. — The problem may be solved by aid of the principle of Art. 107, as follows:

Let the two vertical forces of 5 lbs. and 20 lbs. be replaced by their resultant, R, a downward force of 25 lbs. whose line of action divides BD into segments of lengths $a/5$ and $4a/5$ (Art. 86). Let this line of action intersect BC at E (Fig. 55); then the force P acting upon the bar at A must act through the point E (Art. 107), and its line of action is therefore AE. The three forces P, Q and R are in equilibrium, hence their vector sum is zero. Representing R by the vector FG, and drawing GH parallel to EC and FH parallel to EA, the vectors GH and HF must represent Q and P in magnitude and direction.

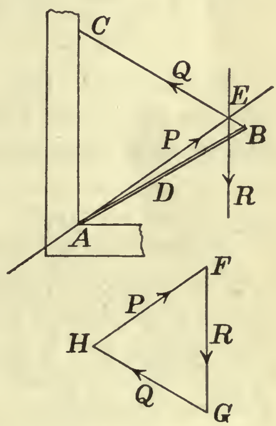


FIG. 55.

The magnitudes P and Q may be found from the geometrical relations, thus: The triangle ABC being equilateral, we have (Fig. 55)

$$CA = BC = 2a;$$

$$BE = BD/5 = a/5;$$

$$EC = 2a - a/5 = 9a/5.$$

Since CAE and FGH are similar triangles, we have

$$GH/FG = EC/CA,$$

or $$Q/R = (9a/5) \div (2a) = 9/10;$$

$$\therefore \quad Q = 0.9\, R = 22.5 \text{ lbs.}$$

Also
$$\begin{aligned} P^2 &= Q^2 + R^2 - 2QR \cos 60^\circ \\ &= (22.5)^2 + (25)^2 - 2 \times 22.5 \times 25 \times 0.5 \\ &= 568.75; \\ P &= 23.85 \text{ lbs.} \end{aligned}$$

To find the direction of P, we have (θ being, as above, the angle between P and the horizontal)

$$\sin\theta = \cos(HFG) = (P^2 + R^2 - Q^2)/2PR = 0.5767;$$
$$\theta = 35^\circ\ 13'.$$

II. A bar AB (Fig. 56) of mass W lbs., whose center of gravity is at any point, rests with the end A on a smooth horizontal plane and the end B against a smooth vertical plane. At A is attached a string which passes over a smooth peg at C and sustains a body of P lbs. Find the position of equilibrium and the pressures exerted upon the bar by the smooth planes.

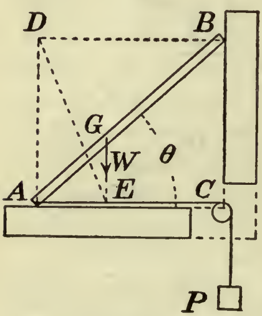


FIG. 56.

Solution.—The center of gravity of the bar being at G, let $AG = a$, $BG = b$ (known quantities), and let angle $BAC = \theta$ (unknown). Following the above outline we have:

(1) The conditions of equilibrium are to be applied to the bar AB.

(2) The forces acting upon the bar are four: its weight W lbs., acting vertically downward at G; the horizontal pull of P lbs. exerted by the string at A; the pressure of the plane at A, directed vertically upward, its unknown magnitude being called R; the pressure of the plane at B, directed horizontally, its unknown magnitude being called S.

(3) For the equations of equilibrium, let three moment equations be written as follows:

With origin at A,

$$S(a+b)\sin\theta - Wa\cos\theta = 0. \qquad (1)$$

With origin at B,

$$P(a+b)\sin\theta - R(a+b)\cos\theta + Wb\cos\theta = 0. \qquad (2)$$

With origin at C,

$$S(a+b)\sin\theta - R(a+b)\cos\theta + Wb\cos\theta = 0. \qquad (3)$$

(4) The unknown quantities are R, S and θ, three in number; hence the three equations are sufficient.

(7) By inspection of equations (2) and (3) it is seen that

$$S = P.$$

From equation (1),

$$\tan\theta = Wa/S(a+b) = Wa/P(a+b).$$

From equation (2),

$$R = P\tan\theta + Wb/(a+b) = W.$$

[If two of the equations of equilibrium had been those obtained by resolving forces horizontally and vertically, the relations $R = W$ and $S = P$ would have appeared at once. The above method was chosen to illustrate the sufficiency of three moment equations.]

Geometrical solution.—Representing by Q the resultant of P and W, the line of action of Q must pass through E (Fig. 56). Again, the line of action of the resultant of R and S must pass through D. These resultants must be equal and opposite and act in the same line DE. We have therefore

$$DA/AE = W/P;$$

or since $\quad AE = a\cos\theta$ and $DA = (a+b)\sin\theta$,

$$\tan\theta = Wa/P(a+b).$$

Also, since the resultant of R and S is equal and opposite to Q, and since S is parallel to P and R parallel to W, the triangle of forces shows that $R = W$ and $S = P$. The triangle of forces is not shown, but should be drawn by the student.

III. A straight bar AB (Fig. 57), whose center of gravity is at its middle point, rests with one end B against a smooth vertical wall

and the other end upon a smooth horizontal floor. A string is attached to the bar at A and is fastened to the wall at C. The lengths AB, AC and AD being given, discuss the possibility of equilibrium.

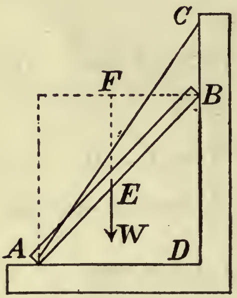


FIG. 57.

Solution.— Let $AB = l$; $AC = a$; $AD = b$; angle $BAD = \alpha$; angle $CAD = \beta$. Following the outline given in Art. 108 we proceed as follows:

(1) The body whose equilibrium is to be considered is the bar AB.

(2) The forces acting upon the bar are four: its weight W lbs. acting vertically downward at its middle point; the pressure (P lbs., unknown) exerted by the wall at B, its direction being horizontal since the wall is smooth; the pressure (Q lbs., unknown) exerted by the floor at A, its direction being vertically upward; the force (R lbs., unknown) exerted by the string at A, its line of action being AC.

(3) Resolving forces horizontally and vertically, and taking moments about A, we obtain the following three equations:

$$-P + R\cos\beta = 0; \qquad (1)$$

$$Q + R\sin\beta - W = 0; \qquad (2)$$

$$Pl\sin\alpha - \tfrac{1}{2}Wl\cos\alpha = 0. \qquad (3)$$

(4) The unknown quantities are only three, P, Q and R, equal in number to the equations, aside from the angles α and β, whose values are easily expressed in terms of l, a and b.

(5) The geometrical relations are merely the values of $\cos\alpha$ and $\cos\beta$ in terms of the known lengths. They are

$$\cos\alpha = b/l; \quad \cos\beta = b/a.$$

(7) To solve the equations, proceed as follows: From equation (3),

$$P = \tfrac{1}{2}W\operatorname{cotan}\alpha = \frac{Wb}{2\sqrt{l^2 - b^2}}.$$

From equation (1),

$$R = \frac{P}{\cos\beta} = \frac{Pa}{b} = \frac{Wa}{2\sqrt{l^2 - b^2}}.$$

From equation (2),

$$Q = W - R \sin \beta = W \left[1 - \frac{\sqrt{a^2 - b^2}}{2\sqrt{l^2 - b^2}} \right].$$

The unknown forces are thus completely determined.

A closer analysis shows, however, that in certain cases the solution fails. The force P cannot act toward the right, and the force Q cannot act downward. If, therefore, either of the above values of P and Q becomes negative, the solution fails. In such a case equilibrium is impossible. The value found for P will always be positive; but Q will be negative if

$$a^2 - b^2 > 4(l^2 - b^2),$$

that is if

$$a^2 > 4l^2 - 3b^2.$$

Numerical case.— Suppose $AB = l = 8$ ft.; $AC = a = 10$ ft.; $AD = b = 6$ ft. Then

$$P = \frac{Wb}{2\sqrt{l^2 - b^2}} = \frac{6}{2\sqrt{28}} W = 0.567\ W;$$

$$R = \frac{Wa}{2\sqrt{l^2 - b^2}} = \frac{5}{2\sqrt{7}} W = 0.945\ W;$$

$$Q = W \left[1 - \frac{\sqrt{a^2 - b^2}}{2\sqrt{l^2 - b^2}} \right] = W \left[1 - \frac{2}{7}\sqrt{7} \right] = 0.244\ W.$$

This value of Q being positive, equilibrium is possible. This might be foreseen from the condition above deduced; for

$$a^2 = 100;$$

$$4l^2 - 3b^2 = 148;$$

hence

$$a^2 < 4l^2 - 3b^2.$$

If, however, we take $b = 7.5$ ft., the other data remaining unchanged, the value of Q is negative, and the solution fails.

Geometrical analysis.—The resultant of P and W acts through the intersection of their lines of action (F, Fig. 57); hence for equilibrium the resultant of Q and R must act through F. Since Q must act upward, the resultant of Q and R acts in some line between AC and the vertical; hence this line cannot pass through F unless F lies

above AC. In the limiting case in which AC passes through F, we have, from the geometrical relations,

$$a^2 = 4l^2 - 3b^2.$$

Hence if $a^2 > 4l^2 - 3b^2$, equilibrium is impossible. This agrees with the result reached above.

EXAMPLES.

1. A bar of mass W lbs. and length l rests with one end upon a smooth horizontal floor and the other against a smooth vertical wall, being held in equilibrium by a horizontal string attached at a point distant b from the upper end. The center of gravity is at a distance a from the upper end, and the inclination to the horizontal is θ. Determine completely all forces acting upon the bar.

Ans. Tension in string $= W(l - a)/b \tan\theta$.

2. In Ex. 1, let $W = 20$ lbs., $\theta = 40°$, $a = 0.6l$, $b = 0.3l$; determine all forces completely.

Ans. Reaction at lower end $= 20$ lbs.; at upper end, 31.79 lbs.; tension $= 31.79$ lbs.

3. A bar of mass 15 lbs., whose center of gravity is at its middle point, rests with its ends upon two smooth planes inclined to the horizontal at angles of 36° and 45° respectively. Determine the inclination of the bar to the horizontal when in equilibrium; also the pressures exerted upon it by the supporting planes.

Ans. 10° 39′; 8.93 lbs.; 10.74 lbs.

4. A bar of known mass and length being supported as in the preceding example, let its center of gravity be at any point of its length, and let α and β be the angles made by the planes with the horizontal. Determine the position of equilibrium and the supporting pressures.

5. A person weighing 160 lbs., standing upon a scale platform, pulls upon a suspended rope in a direction inclined 10° to the vertical. The scale beam shows his apparent weight to be 140 lbs. With what force does he pull the rope? *Ans.* 20.3 lbs.

6. A bar AB, 16 ft. long, is supported in a horizontal position by a smooth hinge at A and a smooth plane at B, the inclination of the plane to the horizontal being 20°. Weights of 20 lbs., 18 lbs. and 28 lbs. are suspended from the body by cords attached at points whose distances from A are 6 ft., 10 ft. and 15 ft. Determine the supporting forces exerted by the plane and hinge. (The pressure exerted by the hinge may have any direction, but always acts through the center of the hinge. See Art. 42.)

Ans. Pressure at $B = 47.89$ lbs.; pressure at $A = 26.63$ lbs.

7. A bar whose mass, center of gravity and length are known, is supported by strings attached to the ends. In order that the bar may rest in a horizontal position, what condition must be satisfied by the directions of the strings? Solve geometrically.

8. A bar AB whose center of gravity is at a distance from the end A equal to one-third the length, is supported as in the preceding example. If the string at A is inclined 30° to the vertical, what must be the inclination of the string at B? Determine the tensions in the strings, the mass of the bar being W lbs.

Ans. 49° 7′; 0.77 W; 0.509 W.

9. A bar AB, whose center of gravity is at its middle point, and whose mass is 12 lbs., is supported in a horizontal position by strings attached to the ends, and sustains loads of 16 lbs. and 20 lbs. at A and B respectively If the string at A is inclined 45° to the horizontal, what is the direction of the string at B? What tensions are sustained by the strings?

Ans. 49° 47′ from horizontal; 31.12 lbs.; 34.05 lbs.

10. A bar is supported at a given angle with the horizontal by strings attached at the ends. Show geometrically the relations that must be satisfied by the directions of the strings. Deduce also an algebraic expression for the relation between the angles made by the strings and the bar with the horizontal.

Ans. Let a and b = segments into which the length of the bar is divided by the center of gravity; θ, α, β = angles made with horizontal by the bar and strings. Then

$$a \tan \alpha - b \tan \beta = (a + b) \tan \theta.$$

11. A bar of mass 20 lbs., whose center of gravity is at its middle point, is supported as in the preceding example, its inclination to the horizontal being 30°. The cord attached at the lower end being inclined 60° to the horizontal, determine the inclination of the other cord, and the tensions in the two cords.

Ans. $\theta = 70° \ 54'$; tensions = 13.23 lbs. and 8.66 lbs.

12. A bar 6 ins. long, whose mass is 2 lbs. and whose center of gravity is 4 ins. from one end, rests inside a smooth hemispherical bowl of radius 4 ins. What is its inclination to the horizontal when in equilibrium, and what are the pressures upon its ends?

110. Equilibrium of Connected Bodies.— If two rigid bodies in equilibrium are connected in any way, as by a smooth hinge or by simple contact, the conditions of equilibrium may be applied to each separately. The forces which the bodies exert upon each other will enter the equations as unknown quantities; and are related to each other in accordance with Newton's third law (Art. 35). That is, A and B being any two bodies, the force which A exerts upon B is

equal, opposite and collinear with the force which B exerts upon A. This will be illustrated by the following problems.

I. Two heavy bars AC and BC (Fig. 58), connected by a smooth hinge at C, are attached to a fixed body by smooth hinges at A and B. Determine the pressures exerted upon the bars by the hinges A and B, and the pressure of each upon the other at C.

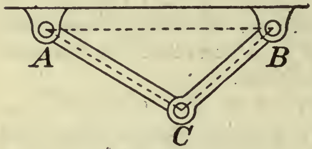


FIG. 58.

Solution.— In the triangle ABC, denote the three angles by A, B, C, and the corresponding opposite sides by a, b, c; and let c be horizontal. Let the center of gravity of each bar be at its middle point.

Let W' = weight of AC;
W'' = weight of BC;
P' = resultant pressure upon AC at A;
H' = horizontal component of P';
V' = vertical component of P';
P'' = resultant pressure upon BC at B;
H'' = horizontal component of P'';
V'' = vertical component of P'';
P = pressure exerted upon AC by BC at C;
($-P$ = pressure exerted upon BC by AC at C;)
H = horizontal component of P;
V = vertical component of P.

(The positive direction is toward the right for horizontal forces and upward for vertical forces.)

Three independent equations of equilibrium for AC may be obtained as follows:

Resolving horizontally,

$$H' + H = 0. \qquad (1)$$

Resolving vertically,

$$V' + V - W' = 0. \qquad (2)$$

Taking moments about A,

$$Vb \cos A + Hb \sin A - \tfrac{1}{2} W'b \cos A = 0. \qquad (3)$$

Three independent equations of equilibrium for BC may be obtained in a similar manner:

$$H'' - H = 0. \quad . \quad . \quad . \quad . \quad (4)$$

$$V'' - V - W'' = 0. \quad . \quad . \quad . \quad (5)$$

$$Va \cos B - Ha \sin B + \tfrac{1}{2} W''a \cos B = 0. \quad . \quad (6)$$

These six equations suffice to determine the six quantities H, V, H', V', H'', V''. The solution may proceed as follows:

From (3) and (6), H and V can be found; the remaining unknown quantities can then be determined at once from the other four equations. The results are as follows:

$$H = \frac{W' + W''}{2(\tan A + \tan B)};$$

$$V = \frac{W' \tan B - W'' \tan A}{2(\tan A + \tan B)};$$

$$H' = -\frac{W' + W''}{2(\tan A + \tan B)};$$

$$V' = \frac{W'}{2} + \frac{(W' + W'') \tan A}{2(\tan A + \tan B)};$$

$$H'' = \frac{W' + W''}{2(\tan A + \tan B)};$$

$$V'' = \frac{W''}{2} + \frac{(W' + W'') \tan B}{2(\tan A + \tan B)}.$$

II. A heavy bar, AB (Fig. 59), is supported at A by a smooth hinge and leans against a smooth circular cylinder which rests in the angle between a horizontal floor and a vertical wall. Determine completely all forces acting upon the bar and upon the cylinder.

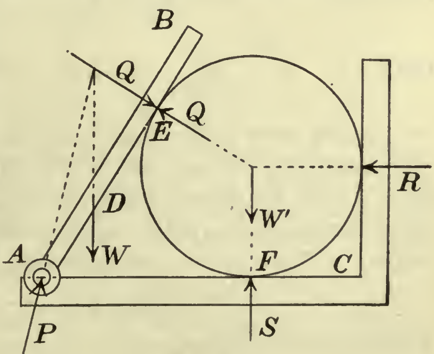


FIG. 59.

Solution.—Let the known data be the weight of the bar, W lbs.; the weight of the cylinder, W' lbs.; the distance of the center of gravity of the bar from A, a ft.; the radius of the cylin-

der, r ft.; the distance AC, b ft. The angle BAC is fixed by the known data; in fact, if $BAC = \theta$,

$$\tan(\theta/2) = r/(b - r).$$

The unknown forces are the pressure on the bar at A, unknown in magnitude and direction; the pressure upon the bar at its point of contact with the cylinder (and the equal and opposite pressure upon the cylinder), known in direction but not in magnitude; and the pressures upon the cylinder at its points of contact with the floor and wall, known in direction but not in magnitude.

Let Q denote the magnitude of the force exerted by the cylinder upon the bar at E; P the magnitude of the force exerted upon the bar by the hinge at A; H and V the horizontal and vertical components of P. Three independent equations of equilibrium for the bar may be written as follows:

Taking moments about A, remembering that $AE = AF = b - r$, we have

$$Q(b - r) - Wa\cos\theta = 0. \quad . \quad . \quad . \quad (1)$$

Resolving horizontally,

$$H - Q\sin\theta = 0. \quad . \quad . \quad . \quad . \quad (2)$$

Resolving vertically,

$$V + Q\cos\theta - W = 0. \quad . \quad . \quad . \quad (3)$$

These three equations suffice for the determination of Q, H and V

Thus, from (1), $$Q = \frac{Wa\cos\theta}{b - r}.$$

From (2), $$H = Q\sin\theta = \frac{Wa\sin\theta\cos\theta}{b - r}.$$

From (3), $$V = W - Q\cos\theta = W\left[1 - \frac{a\cos^2\theta}{b - r}\right].$$

Turning now to the cylinder, let R denote the horizontal pressure exerted by the wall and S the vertical pressure exerted by the floor. Since the lines of action of the four forces acting upon the cylinder intersect in a point, only two independent equations can be written. These may be the following: Resolving horizontally,

$$Q\sin\theta - R = 0. \quad . \quad . \quad . \quad . \quad (4)$$

Resolving vertically,

$$-Q\cos\theta + S - W' = 0. \quad . \quad . \quad . \quad (5)$$

From (4),
$$R = Q \sin \theta = \frac{Wa \sin \theta \cos \theta}{b - r}.$$

From (5),
$$S = W' + Q \cos \theta = W' + \frac{Wa \cos^2 \theta}{b - r}.$$

Geometrical solution.—The lines of action of the three forces acting upon the bar (P, Q and W) must meet in a point; this point is found by prolonging the known lines of action of W and Q. Since W is known, the force-triangle for the three forces can be drawn, and is shown in Fig. 60. The forces P and Q are thus determined.

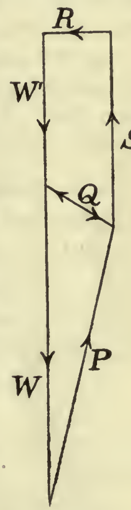


FIG. 60.

The forces acting upon the cylinder are four in number: a force of magnitude Q, opposite to the force Q acting upon the bar; a force W' completely known; forces R and S, whose lines of action are known. The four being in equilibrium, their force-polygon must close, and can be completely drawn from the data now known. It is the quadrilateral whose sides are marked W', Q, S and R in Fig. 60.

In any numerical case, the figure may be drawn to scale and the magnitudes of the required forces found by measurement from the figure. Or, the angles may all be determined from the given data, and the lengths of the sides representing the magnitudes of the required forces may be computed trigonometrically.

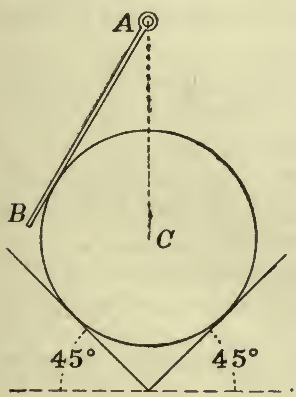


FIG. 61.

EXAMPLES.

1. In Fig. 59, let $AC = 20$ ins., radius of cylinder $= 8$ ins., $AD = 7$ ins., $W = 12$ lbs., $W' = 20$ lbs. Determine all the forces acting upon the bar and cylinder (a) algebraically and (b) geometrically.

2. Find completely the forces acting upon the bar and cylinder in Fig. 61, the bar being hinged at A, and the surfaces of contact all being smooth. Let

the weight of the bar be 10 lbs.; the weight of the cylinder 15 lbs.; the radius of the cylinder 12 ins.; CA (vertical) 24 ins.; distance of center of gravity of bar from A, 8 ins.

Ans. Pressure between bar and cylinder = 1.924 lbs.

3. Three uniform bars AB, BC, CD, of equal length and mass, are connected by smooth hinges at B and C. AB is hinged to a fixed support at A, and CD to a fixed support at D. AB and CD are horizontal and BC is vertical, the system being held in equilibrium by a vertical string attached to AB at a point two-thirds the distance from A to B. Determine all forces acting upon each bar.

4. Two uniform bars, equal in length and mass, connected by a smooth hinge, rest upon a smooth cylinder whose axis is horizontal. Determine the position of equilibrium, and the magnitude and direction of every force acting upon each bar.

111. Simple Machines. — It is often desired to exert upon some body a force of such magnitude or direction that means are not available for applying it directly. In such a case it may be possible to accomplish the desired object by indirect means. Any body or system of connected bodies by which such an object is accomplished is called a *machine*.*

The bodies constituting a machine are usually either actually or nearly in equilibrium. The relations subsisting among the applied forces may therefore be determined from the principles of Statics.

A machine is made to accomplish its object by applying to it forces tending to give it a certain definite motion. A force which aids this motion is called an *effort*, and a force which opposes it a *resistance*. A force which merely guides the motion without any tendency to accelerate or to retard it, is a *constraint*.

The equal and opposite reaction to the force which the machine is designed to exert is a resistance; and since the overcoming of this resistance is the primary object of the machine, it is called a *useful resistance*. The useful resistance may be called the *load*. To determine the relation between the effort and the load is the fundamental statical problem presented by a machine.

Besides the useful resistance there are other forces which act in opposition to the effort and diminish the effectiveness of the machine

---

* Machines considered as devices for the transmission of energy are treated in Chapters XVII and XXIII.

in overcoming the useful resistance. These may be called *wasteful resistances.* The most important wasteful resistances are the frictional forces exerted by the bodies by whose agency the motion of the machine is guided in the desired manner. In the present discussion friction will be disregarded. The method of estimating its effect will be considered in Chapter VII.

A machine is said to yield a *mechanical advantage* if the load is greater than the effort. If the load is less than the effort there is a *mechanical disadvantage.*

The method of applying the principles of equilibrium to machines is illustrated by the following simple examples. Other examples are given in later chapters.

Lever.—A lever is a rigid bar, movable about a fixed axis. The effort and load may be applied at any points of the bar and in any directions, but are usually taken as acting in a plane perpendicular to the fixed axis.

The fixed axis or support about which the lever turns is called the *fulcrum.* The portions of the bar between the fulcrum and the points of application of the effort and the load are called the *arms.*

The lever may be either *straight* or *bent;* and the forces applied to it may or may not be parallel.

The application of the principles of equilibrium to the lever is so simple, when friction is neglected, that the solution of the problem will be omitted.

Fixed Pulley.—A pulley consists of a wheel which can rotate freely about an axis, together with a flexible cord wrapped about some part of the circumference of the wheel. The axis about which the wheel revolves may or may not be stationary; so that pulleys may be classed as *fixed* and *movable.*

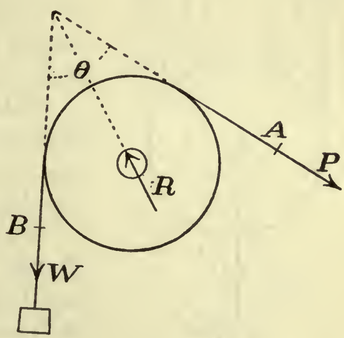


FIG. 62.

Fig. 62 represents a fixed pulley, the effort and load being applied to the same cord, passing around a portion of the circumference of the wheel. The relation between the effort (P) and the load (W) may be found by applying the principles of equilibrium to the

system of bodies * consisting of the wheel and the adjacent portion of the cord. At A there is a tension in the cord equal to P, and at B a tension equal to W. The only other force acting upon the system considered is that exerted by the axle about which the wheel revolves; and if there is no friction between the wheel and the axle, this force must act in a line through the center of the wheel. Taking moments about this center, we have (calling a the radius of the wheel)

$$Pa = Wa; \text{ or } P = W.$$

Let R be the pressure exerted by the axle upon the wheel; then R must be equal and opposite to the resultant of P and W. Its line of action therefore bisects the angle between the two straight portions of the cord; or, calling this angle θ, we have

$$R = 2W \cos(\theta/2).$$

Evidently there is no mechanical advantage in this case; the fixed pulley merely serves as a means of changing the direction of application of a force.

Movable Pulley.—A simple movable pulley is represented in Fig. 63. The relation between the effort (P) and the load (W) is determined by considering the system consisting of the wheel and the adjacent portions of the two cords, limited by the points A, B and C. Neglecting friction and the weight of the pulley, the principles of equilibrium lead to the following results:

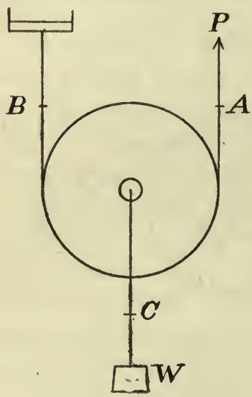


FIG. 63.

Taking moments about the center of the pulley, it is seen that the tensions in the cord at A and B are equal and each equal to P.

Resolving forces vertically, it is seen that the tension in the cord at C (which is equal to W) must be equal to the sum of the tensions at A and B, or $2P$; that is,

$$P = W/2.$$

---

* Although we have thus far applied the principles of equilibrium only to rigid bodies, it is allowable to treat as rigid any body or connected system of bodies, every part of which is in equilibrium. See Chapter VI.

Systems of Pulleys.—By combining fixed and movable pulleys in various ways, the mechanical advantage may be made very great, being limited only by the prejudicial resistances due to friction and the lack of perfect flexibility of the cords.

Fig. 64 shows a system very commonly employed in raising heavy weights. Two sets of wheels are employed (shown at A and B), each set consisting of several pulleys mounted on a common axis but revolving independently of one another. One set (A) is attached to a fixed support, while to the other set, which is movable, the load is applied.*

Neglecting friction and the rigidity of the cord, the relation between P and W is easily obtained. The tension has the same value in all parts of the cord, as may be seen by applying the principle of moments to each wheel separately, taking origin at the center. If n is the number of "plies," or portions of rope supporting the lower set of pulleys, we have (neglecting the weight of the pulleys and of the cord, and assuming that the straight portions of the cord are all parallel)

$$nP = W; \quad P = W/n.$$

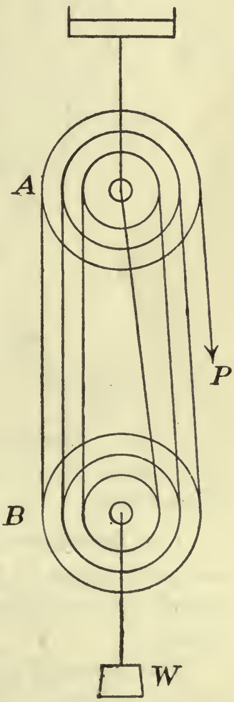


FIG. 64.

EXAMPLES.

1. The bar AB (Fig. 65) is free to rotate about a smooth pin at C. The effort P and the resistance W act at angles θ and ϕ with

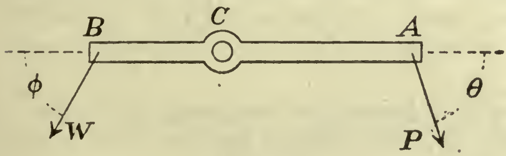


FIG. 65.

* In the system of pulleys as actually constructed, the wheels of each set are usually of equal diameter and mounted side by side upon the same axle. The different portions of the cord are therefore not all coplanar. Each of the separate bodies (each pulley and the block carrying the pulleys) is however acted upon by forces which are practically coplanar.

the bar. Determine the relation between P and W, and compute the magnitude and direction of the force exerted upon the bar by the pin.

2. In Ex. 1, let $AC = 3$ ft., $CB = 2$ ft., $\theta = 60°$, $\phi = 30°$, $W = 85$ lbs. Determine P and the magnitude and direction of the pressure exerted upon the pin.

Ans. $P = 32.7$ lbs.

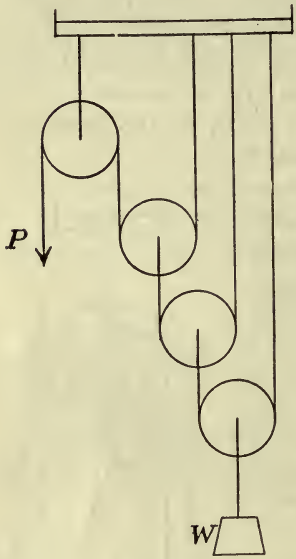


FIG. 66.

3. If P and W (Fig. 62) act in directions inclined 90° to each other, what is the pressure on the axle?

4. In case of a movable pulley, let the effort act at an angle α with the horizontal. What must be the direction of the cord between the pulley and the fixed point of attachment of the cord? What is the relation between P and W?

Ans. $P = W/2 \sin \alpha$.

5. Determine the relation between P and W (Fig. 63) taking account of the weight of the pulley.

6. Determine the relation between P and W in case of a system such as shown in Fig. 64, the number of movable pulleys being 3.

7. In Fig. 64, what must be the strength of the cord if a weight of 5,000 lbs. is to be raised?

8. In Fig. 66, let the weight W be 1,000 lbs. Determine the tension in each cord and the value of P.

9. Solve Ex. 8, taking into account the weights of the pulleys, 5 lbs. each.

112. Balance.— Since the weights of bodies in the same locality are proportional to their masses, the mass of any body may be determined by comparing its weight with that of a body whose mass is known.

The essential parts of a *balance* for comparing the weights of bodies are shown in Fig. 67. The beam L can turn freely in a vertical plane about a fixed point C. At A and B are suspended the "scale-pans" M and N, upon which are placed the bodies whose weights are to be compared. The center of gravity of the whole apparatus * is at some point D. When the scale-pans are vacant,

* In determining this center of gravity, the scale-pans are to be regarded as particles located at their points of suspension A and B.

the point D will be vertically below C in the position of equilibrium. In this position let CA and CB make equal angles with the horizontal, and let the lengths AC and CB be each equal to a. If bodies of equal mass (and weight) are placed in the two scale-pans, the position of equilibrium will be unchanged.

Let a body of mass P be placed at M, and a body of mass Q at N, P being greater than Q, and let the beam assume a position of equilibrium, θ being the angle between CD and the vertical in this position. If initially the lines CA and CB make with the horizontal the angle α, their inclinations to the horizontal in the new position

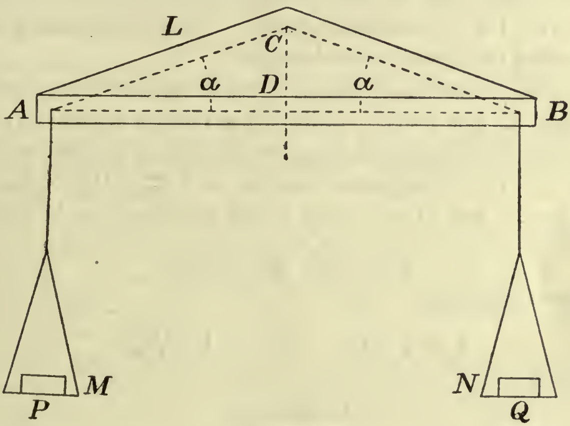


FIG. 67.

of equilibrium are $\alpha + \theta$ and $\alpha - \theta$ respectively. Let $CD = h$, and let $G =$ weight of balance. Taking moments about C,

$$Pa \cos (\alpha + \theta) - Qa \cos (\alpha - \theta) - Gh \sin \theta = 0.$$

Solving for θ,

$$\tan \theta = \frac{(P - Q)a \cos \alpha}{(P + Q) a \sin \alpha + Gh}.$$

In order that a small difference between P and Q may be detected with certainty, a small value of $P - Q$ should cause a relatively great value of θ. This sensitiveness depends upon the values of α, h and a. For given values of a and α, the sensitiveness increases as h decreases. For given values of a and h, the sensitiveness increases as α decreases.

A very high degree of sensitiveness can be obtained only at the expense of stability. That is, if a very small value of $P - Q$ produces a large value of θ, the beam cannot readily be brought into a condition of equilibrium, but will oscillate between positions far from the position of rest.

In the practical use of the balance, the body whose mass is to be determined is placed in one scale-pan, and standard "weights" (bodies of known mass) are placed in the other in sufficient quantity so that the position of equilibrium is the same as when both scale-pans are vacant. In order that the indications of the balance may be correct when used in this manner, the horizontal projections of AC and BC (Fig. 67) must be equal. These horizontal projections may be called the *arms* of the balance.

If the arms are unequal, this fact may be detected by interchanging the weights in the pans. Thus, let a body whose true weight is x lbs. be balanced against standard weights in a balance whose arms are a and b. Let the apparent weight of x be P lbs. when it acts with the arm a, and Q lbs. when it acts with the arm b. Then

$$xa = Pb; \quad xb = Qa.$$

From these equations,

$$x = \sqrt{PQ}; \quad a/b = \sqrt{P/Q}.$$

Examples.

1. In Fig. 67, let the distances AC and CB be each 15 ins., CD 6 ins., the angles ACD and BCD each 80°, and the weight of the balance 4 lbs. What is the position of equilibrium if weights of 10 lbs. and 10 lbs. + ⅛ oz. are placed in the two scale-pans?

Ans. $\theta = 0° \; 5' +$.

2. A body appears to weigh 18 oz. when placed in one scale-pan and 18½ oz. when placed in the other. What is its true weight, and what is the ratio between the lengths of the arms?

Ans. 18.248 oz.; 0.986.

Miscellaneous Examples.

1. A bar AB rests with the end A upon a smooth horizontal plane, and leans against a smooth cylindrical peg at a given distance above the plane. It is held from slipping by a horizontal string AC attached at a fixed point C. Determine all forces acting upon the bar.

2. In Ex. 1, let the height of the peg above the plane be 18 ins., the distance of the center of gravity of the bar from A 12 ins., the inclination of the bar to the plane 25°, its weight 35 lbs. Determine all forces acting upon the bar.

Ans. Vertical pressure at $A = 26.90$ lbs.; pressure of peg $= 8.94$ lbs.; tension $= 3.78$ lbs.

3. Let the data be as in Ex. 1, except that the horizontal string AC passes over a smooth pulley at C and sustains a body of known weight. Determine the position of equilibrium and all forces acting upon the bar. [The solution leads to a cubic equation.]

4. Solve Ex. 3 with the following numerical data: Weight of bar, 10 kilogr.; weight of suspended body, 2 kilogr.; height of peg above plane, 50 c.m.; distance of center of gravity of bar from A, 50 c.m.

Ans. Angle of bar with horizontal $= 77° \ 56'$ or $28° \ 30'$.

5. A uniform bar AB of known mass and length is supported in a horizontal position by a smooth hinge at A and a cord BC inclined to the bar at a known angle and fixed at C. Two bodies of known mass are suspended from the bar at points midway between the center of gravity and the ends A and B respectively. Determine all forces acting on the bar.

6. Solve Ex. 5 with the following numerical data: Weight of bar, 18 lbs.; suspended weights, 3 kilogr. and 5 kilogr.; inclination of string to bar, 37° (measured from prolongation of AB).

Ans. Tension $= 14.26$ kilogr.; hinge reaction $= 13.67$ kilogr. inclined $146° \ 20'$ to AB.

7. A bar AB of weight W is supported by a smooth hinge at A and a string attached at B. The string passes over a smooth peg at C and supports a body of weight P. AC is horizontal and equal to AB, and the center of gravity of the bar is at its middle point. Determine the position of equilibrium, and all forces acting on the bar.

$$\textit{Ans.} \ \cos\frac{CAB}{2} = \frac{P}{2W}\left[1 \pm \sqrt{1 + \frac{2W^2}{P^2}}\right].$$

8. Solve Ex. 7, taking $W = 28$ lbs., $P = 6$ kilogr.

Ans. $CAB = 21° \ 54'$ or $241° \ 20'$. Hinge reaction $= 15.25$ lbs. or 36.53 lbs.

9. In Ex. 7, let AC and AB be unequal, AC being still horizontal. Deduce a cubic equation for determining $\cos\theta$, where θ is the angle between the bar and the horizontal.

10. Solve Ex. 9, taking $AC = 2\,(AB)$, $W = P = 12$ kilogr.

Ans. $\theta = 14° \ 55'$, or $215° \ 42'$.

11. Modify Ex. 7 by taking $AC = AB$, but not horizontal.

12. A smooth cylinder of radius a is placed with its axis horizontal and parallel to a smooth vertical wall at distance h. A uniform bar of weight W and length $2l$ rests against the cylinder and

with its lower end against the wall. Determine the position of equilibrium.

Ans. If θ = angle between bar and vertical, $l \sin^3 \theta + a \cos \theta = h$.

13. In Ex. 12, let the radius of the cylinder be very small compared with h and l. Solve with the following data: $W = 15$ kilogr., $h = 6$ ins., $l = 12$ ins. Determine the position of equilibrium and all forces acting on the bar.

14. A uniform smooth bar of length $2l$ and weight W rests in a hemispherical bowl of radius a. Assuming the length to be so great that the upper end projects beyond the edge of the bowl, determine the position of equilibrium and all forces acting on the bar.

Ans. The angle θ between the bar and the horizontal is given by the equation $\cos \theta = (l \pm \sqrt{l^2 + 32a^2})/8a$.

15. Solve Ex. 14 with the following numerical data: $W = 12$ kilogr., $l = 0.45$ met., $a = 0.40$ met.

Ans. $\theta = 30° \, 28'$; pressure at lower end = 7.05 kilogr.

16. A heavy bar is supported by a string attached at any two points and passing over a smooth peg. Determine the position of equilibrium and the tension in the string.

Ans. The peg divides the string into segments whose lengths are directly proportional to the distances of the center of gravity of the bar from the points of attachment of the string.

17. In the preceding example let the weight of the bar be 12 lbs., the distances of the center of gravity from the points of attachment of the strings 2 ft. and 3 ft., the length of the string 7.5 ft. Determine the tension in the string in the position of equilibrium.

Ans. 7.89 lbs.

18. What horizontal force is necessary to pull a carriage wheel over a smooth obstacle, the radius of the wheel being a, its weight W, and the height of the obstacle h?

19. A heavy uniform beam, movable in a vertical plane about a smooth hinge at one end, is sustained by a cord attached to the other end. The angle between the bar and the vertical being fixed, determine what direction of the cord will cause the least pressure on the hinge. Determine the corresponding values of the unknown forces acting on the bar.

Ans. Let α = angle of bar with vertical, θ = angle of string with bar, then $\tan \theta = \frac{1}{2} \tan \alpha$.

20. A heavy beam rests with one end against a smooth inclined plane; to the other end is attached a cord which passes over a smooth pulley and sustains a given weight. Determine the position of equilibrium.

21. The roof-truss shown in skeleton in Fig. 68 is supported by a smooth horizontal surface at A and a smooth hinge at B. It is re-

quired to sustain the wind pressure on a portion of the roof of width 12 ft. and length AC or CB. Assume the pressure to be normal to the roof and uniformly equal to 20 lbs. per sq. ft. Compute the supporting forces at A and B when the wind is from the right; also when it is from the left.

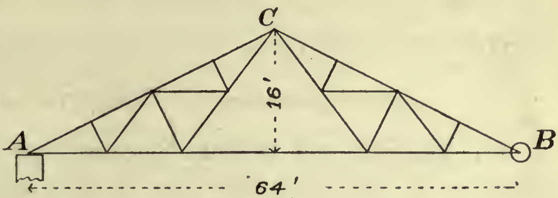


FIG. 68.

22. Take data as in Ex. 17, but assume the peg rough, so that the tensions in the two portions of the string may be unequal. If the bar is in equilibrium when the peg divides the string into segments 1.9 ft. and 5.6 ft. in length, what are the tensions in the two portions of the cord?

Ans. 10.46 lbs. and 4.18 lbs.

23. The lines of action of three forces intersect in points A, B, and C. Prove that, if their resultant is a couple, they are proportional to the vectors AB, BC, CA.

24. A uniform bar of weight W rests with its ends against two smooth planes whose inclinations to the horizontal are α and β, that of the bar being θ. A weight P is suspended from a point distant one-third the length of the bar from one end. Required the value of P for equilibrium.

CHAPTER VI.

EQUILIBRIUM OF PARTS OF BODIES AND OF SYSTEMS OF BODIES.

§ 1. *Equilibrium of Any Part of a Body.*

113. External and Internal Forces Acting on a Body.— The conditions of equilibrium have thus far been applied only to a particle or to a rigid body regarded as a whole. In case two or more bodies were considered, each was regarded as presenting a separate problem in equilibrium. For certain purposes, however, it is desirable to confine the attention to a portion of a rigid body; and for other purposes it is found convenient to treat two or more bodies as together forming a system. As a preliminary to the discussion of problems from these points of view, it is useful to recur to the fundamental conception of force.

By the definition (Art. 32), a force is an action exerted by one body upon another. Again, by a fundamental law (Art. 35), forces always act in pairs ; so that when any one body exerts a force upon another, the second body exerts an equal and opposite force upon the first. Thus every force concerns *two bodies ;* although in the preceding chapters the attention has in every case been directed especially to a single body,— that upon which the force acts,— and the body exerting the force has not always been specified.

The word body, in the definition of force, must be understood to mean *any portion of matter.* The two portions may be separate from each other (as in most of the cases hitherto discussed), or they may be *parts of a single body.* This leads to the following classification of the forces applied to a body:

An *external force* is one exerted upon the body in question by some other body.

An *internal force* is one exerted upon one portion of the body by another portion of the same body.

114. Conditions of Equilibrium for a Rigid Body as a Whole.— The discussions of the composition, resolution and equilibrium of forces, as thus far given, have related to *external* forces applied to the same rigid body, *internal* forces not appearing in the equations or conditions of equilibrium.

Thus, although it is generally true that adjacent portions of a rigid body exert forces upon each other, these forces do not come into consideration when the problem relates solely to the equilibrium of the body as a whole. It may, however, be desired to gain some information regarding these internal forces. It will then be necessary to direct the attention to a *portion* of the body, and to consider the relations among all the forces which act upon this portion.

115. Conditions of Equilibrium for a Portion of a Body.— Since the body to which the conditions of equilibrium apply may be *any portion of matter* whose particles are rigidly connected, it may be *any connected portion* of a rigid body. But in writing the equations of equilibrium for a portion of a body, certain forces must be included which are omitted if the equilibrium of the whole body is considered. These are the forces exerted *upon* the portion in question *by* other portions of the same body. Such forces are *internal* when the whole body is considered, but *external* when one part is considered.

To illustrate, let AB (Fig. 69) represent a bar acted upon by two forces of equal magnitude applied at the ends in opposite directions parallel to the length of the bar. These forces are exerted upon AB by some other bodies not specified. If the whole bar be considered, the *external* forces are the two forces named. But suppose the body under consideration is AC, a portion of AB; the external forces acting upon this body are (1) a force applied at A (already mentioned) and (2) a force applied at C (exerted *upon* AC *by* CB). This latter force is *internal* to the bar AB but *external* to AC. In applying the conditions of equilibrium to AB, the only forces to enter the equations are those exerted at A and B as already mentioned; but if the equations of equilibrium for AC are to be written, the forces to be included are the force at A and the force exerted upon AC by CB at C.

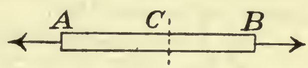


Fig. 69.

§ 2. *Determination of Internal Forces.*

116. General Method of Determining Internal Forces.— Little is in general known regarding the forces which contiguous portions of a body exert upon each other. But when the external forces are

known, the internal forces can be partly determined by applying the conditions of equilibrium to portions of the body. Thus, to take the simplest and most important case, suppose it is known that not only the whole body, but every portion of it, is in equilibrium.* For such cases the following proposition may be stated:

If a body, every part of which is in equilibrium, be conceived as made up of two parts, X and Y, then the internal forces exerted by X upon Y, together with the external forces acting on Y, form a system in equilibrium.

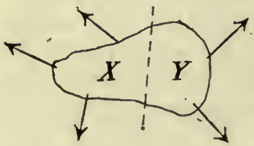


FIG. 70.

Thus, by applying the conditions of equilibrium to the body Y, the resultant of all the forces exerted by X upon Y may be determined; it being equal and opposite to the resultant of the *external* forces applied to Y.

The *individual forces* exerted by the particles of X upon those of Y cannot, however, be determined, these forces being in general very numerous and acting in various unknown directions.

EXAMPLES.

1. A beam 8 ft. long, weighing 40 lbs. per linear foot, rests horizontally upon supports at the ends. Dividing the beam by a transverse plane 3 ft. from one end, determine the resultant of the forces exerted by each portion upon the other.

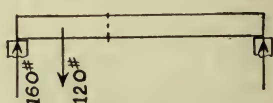


FIG. 71.

The forces acting upon the left portion of the beam (Fig. 71) are (1) the supporting force of 160 lbs.; (2) the weight of three linear feet of the beam, equivalent to a force of 120 lbs. applied 1.5 ft. from the left end; and (3) the forces exerted by the other portion of the beam. The resultant of the first two is an upward force of 40 lbs. acting in a line 4.5 ft. to the left of the left support. Hence the forces exerted by

* It is to be carefully noted that the equilibrium of a body does not necessarily imply the equilibrium of every portion of it. Thus, if the external forces applied to the whole body are balanced, it may still have a motion of rotation, in which case the forces acting upon a portion of the body will in general be unbalanced. This will be apparent when the rotation of a rigid body has been studied. In very many cases, especially such as arise in engineering practice, the bodies are not only in equilibrium, but at rest. In such cases all parts of the body are also in equilibrium.

the other portion of the beam have for their resultant a *downward* force of 40 lbs. acting in a line 4.5 ft. to the left of the left support.

2. A beam 12 ft. long, of uniform density and cross-section, weighing 20 lbs. per linear foot, rests horizontally upon supports at the ends. Conceiving the beam divided by a transverse plane through the middle of the length, determine the resultant of the forces exerted by each portion upon the other.

3. In Ex. 2, let the dividing plane be 3 ft. from one end. Determine the resultant of the forces exerted by each portion upon the other.

4. Let the same beam rest upon a single support at the middle. Determine the resultant of the forces exerted upon each other by the two portions described in Ex. 2; also by the two portions described in Ex. 3.

5. Let the same beam rest upon supports, one of which is 2 ft. from one end and the other 4 ft. from the other end. Answer the questions asked in examples 2 and 3.

6. A uniform beam 17 ft. long, weighing 120 lbs., rests horizontally upon supports at the ends and sustains two loads: a load of 40 lbs. 3 ft. from the left end, and a load of 80 lbs. 6 ft. from the right end. Taking a transverse section 4 ft. from the left end, compute the resultant of the forces exerted by one of the two portions of the beam upon the other.

Ans. 52.94 lbs.; 3.3 ft. from left end and 20.3 ft. from right end.

7. In Ex. 6, let the transverse section be taken 4 ft. from the right end. Answer the same question.

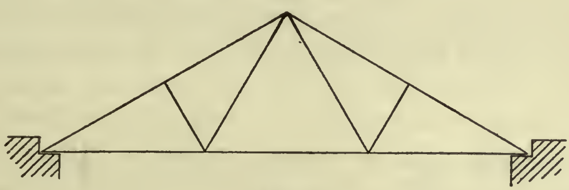


FIG. 72.

117. Jointed Frame.— Engineering practice has to deal with structures made up of straight bars connected at the ends by hinge joints. (See Art. 42.) A structure thus made is called a *jointed frame* or *truss*. The determination of the internal forces in such a frame is an important problem in engineering.

Fig. 72 represents in skeleton a simple form of roof truss, resting upon supports at the ends. Such a truss is acted upon by external forces consisting of the weight of its own members, the weight of

the roof proper, sometimes the weight of snow and the pressure of wind, and the supporting forces at the ends. Each of the bars composing the truss is subjected to certain forces tending to break it; this tendency is resisted by the internal forces called into action between the contiguous portions of the bar. If these internal forces exceed certain values, the material is injured and the bar may be broken; it is therefore important to be able to determine their magnitudes before designing the members.

To solve this problem the general method of Art. 116 is employed. But in order to simplify the problem and make it completely determinate, certain assumptions are made which are only approximately true.

(*a*) *It is assumed that the hinges are without friction.* The effect of this may be seen by referring to Fig. 14 (p. 21), which represents a portion of one bar. The connection with other bars is made by means of a pin A which passes through holes in the ends of the bars. The pin being assumed smooth, the pressure of the pin upon the bar is normal to the cylindrical surfaces of both pin and bar, and therefore acts through the axes of these cylinders.

(*b*) *It is assumed that all external forces acting upon the truss are applied to the pins.* If this is the case, the only forces acting on any bar are those exerted by the two pins at its ends. These two forces being in equilibrium, must be equal and opposite and must have the same line of action, which is therefore the line joining the centers of the hinges. If a transverse section be taken dividing the bar into two parts, the principle of Art. 116 shows that *the resultant force exerted by each part upon the other must have the same line of action as the forces exerted upon the ends of the bar by the pins.*

118. Determination of Internal Forces in Jointed Frame.— The method of determining internal forces in a jointed frame will now be explained by reference to a numerical problem. Fig. 73 shows the dimensions of a roof truss supporting three vertical loads at upper joints, and supported upon horizontal surfaces at the ends. It is required to determine the internal forces in each of the bars. No loads will be considered except those shown. The first step in the solution is

(*a*) *The determination of the supporting forces.*—Assuming these to be vertical, each must be equal to half the total load supported, *i. e.*, to 375 lbs.

(*b*) *Determination of internal forces.*—Let a section be taken, as *MN*, dividing the truss into two parts, *X* and *Y*, and let the conditions of equilibrium be applied to either part, as *X*. This body

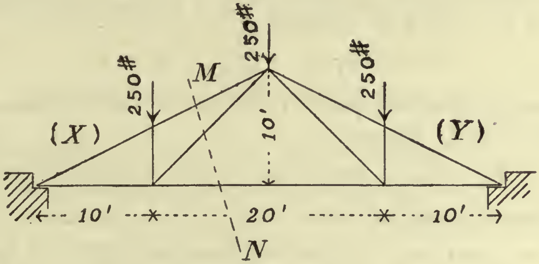


FIG. 73.

X is acted upon by an upward force of 375 lbs. at the support; a downward load of 250 lbs.; and the forces exerted by *Y*. In accordance with the assumptions made in Art. 117, these forces exerted by *Y* are three, acting along the lines of the three members cut by the section *MN*. The magnitudes of these forces are, however, unknown. If three independent equations of equilibrium be written for the body *X*, the only unknown quantities which will enter them are these three unknown force-magnitudes, which may therefore be determined by solving the equations.

Let P_1, P_2 and P_3 denote the magnitudes of the three unknown forces acting upon *X* (Fig. 74), and let them be assumed to act

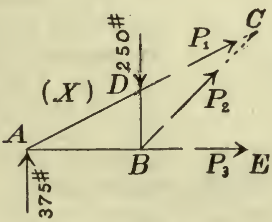


FIG. 74.

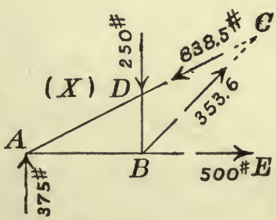


FIG. 75.

away from the body *X*. (Their actual directions will then be known from the signs of their numerical values.) Let the three equations be written as follows:

Taking origin of moments at A, P_1 and P_3 are eliminated. The arm of P_2 is $5\sqrt{2}$ ft., and the equation is

$$P_2 \times 5\sqrt{2} - 250 \times 10 = 0. \quad . \quad . \quad . \quad (1)$$

Next taking origin of moments at B, P_2 and P_3 are eliminated, and the arm of P_1 is $2\sqrt{5}$ ft. Hence

$$-P_1 \times 2\sqrt{5} - 375 \times 10 = 0. \quad . \quad . \quad . \quad (2)$$

Again, taking moments about C, P_1 and P_2 are eliminated, and the arm of P_3 is 10 ft. Hence

$$P_3 \times 10 + 250 \times 10 - 375 \times 20 = 0. \quad . \quad . \quad (3)$$

Solving these equations,

$$P_1 = -838.5 \text{ lbs}; \quad P_2 = +353.6 \text{ lbs}; \quad P_3 = +500.0 \text{ lbs}.$$

The signs of these results show that P_2 and P_3 act *away from* the body X, while P_1 acts *toward* X.

The forces exerted by X upon Y are equal in magnitude to P_1, P_2 and P_3, but opposite in direction.

Since the independent equations of equilibrium may be written in many different ways, the details of the solution may be varied indefinitely. It is often advantageous to use resolution equations instead of moment equations.*

The following examples may be solved by application of the foregoing principles.

EXAMPLES.

1. Determine the internal forces in all the bars of the truss shown in Fig. 73.

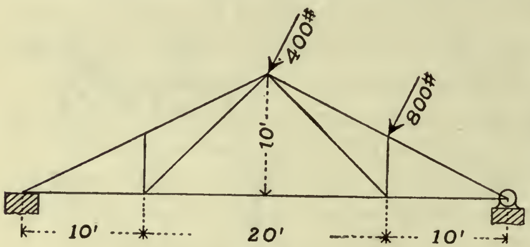


FIG. 76.

2. Let the truss shown in Fig. 76 sustain loads of 400 lbs. and 800 lbs. due to wind pressure, acting normally to the roof and applied at the joints as shown. Suppose the truss

* In the above discussion the aim has been merely to exhibit clearly the general principles upon which the determination of internal forces in framed structures is based. For a full treatment the reader must consult treatises devoted especially to the subject.

supported at one end by a smooth horizontal surface and at the other end by a hinge. Compute the internal forces in all the bars of the truss, neglecting all loads except those specified.

[First determine the supporting forces at the ends by considering the equilibrium of the truss as a whole. Then apply the above method to the computation of the internal forces.]

§ 3. *Equilibrium of a System of Bodies.*

119. External and Internal Forces Acting on a System of Bodies.— For certain purposes it is found convenient to group a number of bodies or particles together in applying the principles of Statics, such a group being called a *system.*

When a system of bodies is considered, a force acting upon any body of the system is called internal or external according as the body exerting the force is or is not a member of the system.

For example, every terrestrial body is attracted by the earth in accordance with the law of gravitation. This attraction is external if the body is considered by itself. If, however, the body and the earth are regarded as forming a system, the force mentioned is internal. Thus, whether the force is external or internal depends upon what has been arbitrarily chosen as the system to be discussed.

120. Conditions of Equilibrium for a System of Bodies.—It will now be shown that if every body in a system is in equilibrium, the *external forces* acting on the system as a whole satisfy the same conditions as if the system were a rigid body in equilibrium.

The equations of equilibrium may be written for each body of the system separately. Let a resolution equation be written for each body, the direction of resolution being the same for all; and let a moment equation be written for each body, the same origin of moments being taken for all. Adding all the resolution equations, the resulting equation shows that

The algebraic sum of the resolved parts of all forces acting upon the bodies is equal to zero, whatever the direction of resolution.

Similarly, the addition of the moment equations shows that

The algebraic sum of the moments of all the forces acting upon the bodies is equal to zero, whatever the origin of moments.

Now, these equations include both external and internal forces. But since the forces exerted by any two of the bodies upon each

other are equal and opposite and have the same line of action, the sum of the resolved parts of two such forces in any direction is zero, and the sum of their moments about any origin is zero. Hence the sum of the resolved parts of all internal forces in any direction is zero, and the sum of the moments of all internal forces about any origin is zero. The following proposition may therefore be stated:

If every body of a system is in equilibrium, the *external forces* applied to the system satisfy the following conditions:

(*a*) *The sum of their resolved parts in any direction is zero.*

(*b*) *The sum of their moments is zero for any origin.*

That is, the external forces satisfy the same conditions as if the system were a rigid body.

The converse of this proposition is, however, not generally true. It cannot be stated that when the external forces for the system satisfy the above conditions, the forces acting on each separate body also satisfy the conditions of equilibrium.

Although the general principle just deduced is often useful in the solution of problems, a complete solution usually requires in addition the employment of some at least of the equations of equilibrium for the bodies taken separately.

121. Application.— Two homogeneous smooth cylinders rest, with axes horizontal, against each other and against two inclined planes making given angles with the horizontal. Required to determine the position of equilibrium.

Regarding the two cylinders as a system, the external forces are the weights of the cylinders (taken as acting vertically through their centers), and the pressures exerted by the supporting planes (acting normally to the surfaces). The pressures exerted by the cylinders upon each other are *internal* to the system.

Two solutions will be given; one partly geometrical, the other algebraic. (See Fig. 77.)

Let r' and r'' denote the radii of the cylinders; W' and W'' their weights; θ the angle between the horizontal and the plane containing the axes of the cylinders; α and β the angles made by the planes with the horizontal; R' and R'' the normal pressures exerted by the planes.

(1) *Geometrical solution.*—Applying the conditions of equilibrium to the external forces acting upon the system, notice that the resultant of W' and W'' is a vertical force of magnitude $W' + W''$,

and is in equilibrium with R' and R''. Since their directions are known, their magnitudes may be determined by means of a triangle.

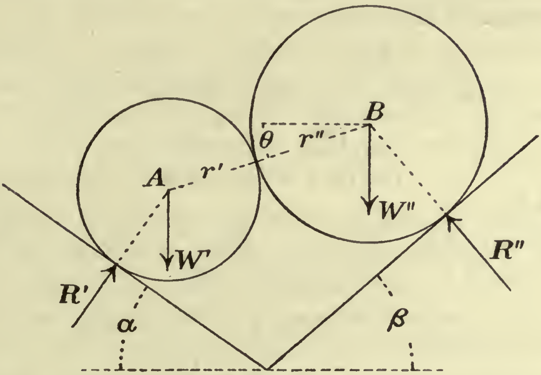


FIG. 77.

Draw LN (Fig. 78) equal and parallel to $W' + W''$, and make NH and HL parallel respectively to R'' and R'; then NH represents R'' in magnitude and direction, and HL represents R'. Also, the angle $HLN = \alpha$, and $HNL = \beta$. Hence

$$\frac{R'}{\sin \beta} = \frac{R''}{\sin \alpha} = \frac{W' + W''}{\sin (\alpha + \beta)},$$

or $$R' = (W' + W'') \frac{\sin \beta}{\sin (\alpha + \beta)},$$

$$R'' = (W' + W'') \frac{\sin \alpha}{\sin (\alpha + \beta)}.$$

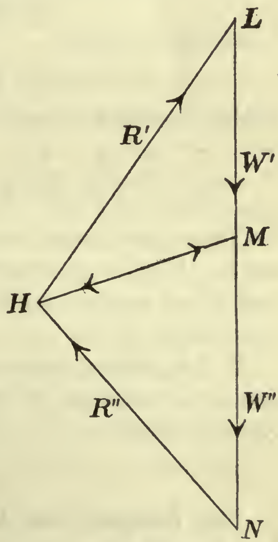


FIG. 78.

Next, to determine θ, notice that the resultant of W' and R' acts in a line through A, and the resultant of W'' and R'' acts in a line through B. Since these two resultants balance each other, each must act along the line AB. Now if, in Fig. 78, the point M be so taken that $LM = W'$ and $MN = W''$, the resultant of R' and W' is represented in magnitude and direction by HM, while MH represents

the magnitude and direction of the resultant of W'' and R''. Hence, for equilibrium, the line AB must be parallel to HM. In the triangle LMH we have angle at $L = \alpha$; angle at $H = 90^\circ - (\alpha + \theta)$; angle at $M = 90^\circ + \theta$;

hence
$$\frac{W'}{\sin(90^\circ - \alpha - \theta)} = \frac{R'}{\sin(90^\circ + \theta)},$$

or
$$\frac{W'}{\cos(\alpha + \theta)} = \frac{R'}{\cos\theta}.$$

Solving for θ,
$$\frac{\cos(\alpha + \theta)}{\cos\theta} = \frac{W'}{R'} = \frac{W'}{W' + W''} \cdot \frac{\sin(\alpha + \beta)}{\sin\beta};$$

or, finally,
$$\tan\theta = \frac{W'' \operatorname{cotan}\alpha - W' \operatorname{cotan}\beta}{W' + W''}.$$

(2) *Algebraic solution.*—Applying the algebraic conditions of equilibrium to the four forces W', W'', R' and R'', three independent equations may be written as follows:

Resolving horizontally,
$$R' \sin\alpha - R'' \sin\beta = 0. \quad . \quad . \quad . \quad (1)$$

Resolving vertically,
$$R' \cos\alpha + R'' \cos\beta - (W' + W'') = 0. \quad . \quad (2)$$

Taking moments about B, and noticing that the arm of R' is $(r' + r'') \cos(\alpha + \theta)$,
$$W'(r' + r'')\cos\theta - R'(r' + r'')\cos(\alpha + \theta) = 0. \quad (3)$$

These three equations contain three unknown quantities, R', R'' and θ. From equations (1) and (2), R' and R'' may be found; and then θ can be determined from (3). The results will agree with those found above.

If it is desired to determine the pressure between the two cylinders, the conditions of equilibrium must be applied to one of the cylinders alone.

§ 4. *Stress.*

122. External and Internal Stresses.—A stress has already been defined as consisting of two equal and opposite forces exerted by two portions of matter upon each other (Art. 36). The two forces of a stress always constitute an "action" and its "reaction."

When any body or system of bodies is under consideration, the stresses with which the system is concerned may be either external or internal. An external stress is one exerted between two portions of matter, one of which belongs to the body or system while the other does not. An internal stress is one exerted between two portions of matter both of which belong to the body or system.

This classification expresses no distinction as to the nature of the stresses. It is merely a distinction made for convenience in dealing with the problems of Mechanics.

123. Kinds of Internal Stress in a Body.— Considering any two adjacent portions of a body, separated by a plane surface, the resultant stress between these two portions may have any direction. Whatever this direction may be, let each of the forces of the stress be resolved into two components, one normal to the surface of separation and the other parallel to it. The normal components of the forces constitute a *normal stress*, and the other components a *tangential stress.*

The normal stress may be either *tensile* or *compressive.* The former resists a tendency of the two portions of the body to separate; the latter resists a tendency of the two portions to move toward each other.

The tangential stress is called a *shearing stress.* It resists a tendency of the two portions of the body to slide past each other along the surface separating them.

To illustrate the two kinds of normal stress, reference may be made to the problem discussed in Art. 118. The roof truss being regarded as consisting of two parts, X and Y, it was found that Y exerts upon X three forces whose magnitudes and directions are shown in Fig. 75; while X exerts upon Y forces equal and opposite to these. It is thus seen that normal stresses act in every transverse section of each bar; the stress in DC being *compressive*, while those in BC and BE are *tensile stresses.* The nature of the normal stress in any bar of a jointed frame may thus readily be found by determining the direction of the internal force exerted by one part of the structure upon the other.

Examples.

1. Determine the kind of stress in each of the bars of the truss shown in Fig. 73, due to the loads described in Ex. 1, Art. 118.

2. In Ex. 2, Art. 118, determine the kind of stress in each member of the truss.

124. Kinds of Stress Between Bodies.— If two bodies are in contact, the forces which they exert upon each other at the surface of contact constitute stresses which may be classified in the same manner as the internal stresses in a body. The names *tension*, *compression* and *shear* are, however, confined to the internal stresses.

The normal stress between two bodies in contact is usually *pressure;* that is, the forces have such directions as to resist a tendency of the bodies to move toward each other. If the bodies for any reason have a tendency to separate, they are in general able to exert only very slight forces to resist this tendency.

The tangential stress between two bodies at their surface of contact can have any magnitude up to a certain limit. Thus, if a body rests upon a horizontal surface, and if a horizontal force is applied to it, the supporting body exerts a force in the opposite direction. If the applied force does not become too great, the opposing force will always just equal it and the body will remain in equilibrium. A *tangential stress* thus acts between the two bodies. The forces composing such a stress are of the kind to which the name *friction* is applied. The subject of friction is treated in Chapter VII.

Besides the stresses acting between bodies at their surfaces of contact, there are other stresses which act between bodies not in contact and without any apparent material connection. These are the so-called "actions at a distance" already mentioned (Art. 40). Such stresses are frequently called *attractions* and *repulsions*, and are analogous to tensile and compressive stresses respectively.

125. Strain.— The external forces applied to a body usually tend to cause the different parts of the body to move relatively to each other. This tendency is opposed by the internal forces called into action. No natural body, however, remains wholly rigid under applied forces, but the parts move, at least slightly, relatively to one another.

Strain is the name given to the deformation which a body undergoes under the action of applied forces.

In this work we shall not usually be concerned with strain, the bodies considered being regarded as rigid. Any ordinary solid body will assume a form of equilibrium under the applied forces (if they

are not such as to cause rupture) and may then be regarded as a strictly rigid body.

EXAMPLES.

1. A straight bar of uniform cross-section and density, 14 ft. long and weighing 120 lbs., rests in a horizontal position upon smooth supports at the ends. Conceive the bar to be divided by a plane perpendicular to the length, 4 ft. from one end. Determine (*a*) the resultant tangential stress and (*b*) the resultant normal stress between the two portions at the surface of separation.

2. With the data of Ex. 1, take a transverse section through the middle of the bar, and determine the resultant tangential stress and resultant normal stress between the two portions.

3. A body weighing 50 lbs. is at rest upon a rough horizontal surface while acted upon by a force of 10 lbs. directed at an angle of 30° upward from the horizontal. Describe completely the resultant stress acting between the given body and the supporting body.

4. A uniform bar AB, 12 ft. long, weighing 18 lbs., is supported at A by a smooth hinge and at B by a string inclined 30° to the vertical. Conceiving the bar divided in the middle by a transverse plane, determine (*a*) the resultant normal stress, (*b*) the resultant tangential stress, and (*c*) the resultant stress, acting between the two portions of the body.

CHAPTER VII.

FRICTION.

126. Smooth and Rough Surfaces.— The conception of a perfectly smooth surface has frequently been employed in the foregoing discussions. Such a surface was defined in Art. 42. The surfaces of actual bodies are, however, always more or less rough.

If any two bodies are in contact, each exerts upon the other a force, the direction of which depends upon various conditions. Let the surface of one of the bodies be a plane, and consider the force P which this body exerts upon the other. Whatever be the direction of this force, let it be resolved into two components N and T, the former perpendicular to the plane surface and the latter parallel to it. (Fig. 79 shows P, N and T in magnitude and direction. The lines of action of P and N are also shown; but T will act along a line lying in the surface of contact.) If the plane surface be very rough, the magnitude of T may be considerable in comparison with N; but if the surface be made smoother, the magnitude of the tangential force which can be exerted becomes smaller. We are thus led, as in Art. 42, to define a perfectly smooth surface as *one which can exert no force parallel to itself upon any body placed in contact with it.* In other words, the resultant pressure exerted by a smooth surface upon any body in contact with it must have the direction of the normal to the surface.

FIG. 79.

Although the surface of a body can never be made to satisfy this definition of perfect smoothness, there are cases which approach near to it. The surface of a sheet of ice is often nearly smooth, and the difficulty of walking on such a sheet is due to the fact that only a small tangential force can be exerted by the ice upon a body in contact with it.

The forces exerted by two bodies upon each other at their surface of contact are of the kind called *passive resistances* (Art. 41). They are called into action to resist a tendency of the two bodies, due to any cause, to move relatively to each other. Thus, suppose a body

to rest upon a horizontal surface and to be acted upon by the attraction of the earth; *i. e.*, by a downward force equal to the weight of the body. The pressure exerted upon it by the table is called into action to resist this downward force and is exactly equal and opposite to it. Let a horizontal force be now applied to the body. If this force be not too great the body will remain at rest, and in order to hold it at rest the supporting body exerts upon it, in addition to the vertical force already mentioned, a force equal and opposite to the horizontal applied force. If the magnitude of the horizontal force be gradually changed from zero to any value (up to a certain limit), or if its direction be changed, the resisting force changes in a corresponding manner so as always to be equal and opposite to it. The resultant force exerted by the supporting body is made up of the upward force resisting the weight of the supported body, and the horizontal force just described. If additional forces be applied to the body, equal and opposite forces will be exerted by the supporting body; but the magnitude and direction of the force that can be resisted are always subject to limitation, depending upon the material composing the bodies and the nature of their surfaces.

127. Friction.— The name *friction* is given to the tangential component of the force exerted by one body upon another at their surface of contact. The frictional forces exerted by the two bodies upon each other are equal and opposite and constitute a stress.

Experiments show that friction acts according to certain laws. In stating these laws, the term "applied forces" will be used to designate all forces acting upon the body in question, except the "passive resistance" exerted by the other body at their surface of contact.

128. Laws of Friction.— The following may be stated as the laws or principles to be employed in discussing the force of friction exerted upon a body by another in contact with it.

(1) If the body is in equilibrium the friction is equal and opposite to the tangential component of the resultant of the applied forces.

(2) If the tangential component of the resultant of the applied forces becomes greater than a certain limiting value, the friction cannot become great enough to balance it. The value of the friction when sliding is about to take place is called the *limiting friction*.

(3) The magnitude of the limiting friction bears a constant ratio to that of the normal pressure between the two bodies.

(4) The ratio of the limiting friction to the normal pressure is

independent of the area of contact of the two bodies, if the touching surfaces are uniform in character.

(5) If the body is sliding along the surface of contact the friction is independent of the velocity and proportional to the normal pressure. In this case, the ratio of the friction to the normal pressure is found to be less than when sliding is about to occur.

Of these five laws, (1) and (2) are exact. The first is, indeed, a direct result of the principles of equilibrium. Laws (3), (4) and (5) are only approximate, but are nearly enough true for most practical applications.

129. Coefficient of Friction.— The ratio of the magnitude of the limiting friction to that of the normal pressure between the bodies is called the *coefficient of friction.* That is, if μ is the coefficient of friction,

$$\mu = F/N,$$

in which F denotes the limiting friction and N the normal pressure.

Law (3) states that μ is constant for two given bodies, whatever the magnitude of N. If the pressure between the two bodies becomes nearly great enough to crush the material of which either is composed, it is found that μ is considerably greater than for small pressures. Within certain limits of normal pressure, however, its value is nearly constant, and it will be regarded as constant in the applications which follow.

It is to be noted that the value of μ depends upon the nature of both the bodies concerned.

130. Angle of Friction.—The total pressure exerted by either body upon the other is the resultant of the normal pressure and the friction. If this resultant be determined by the triangle of forces it is seen that its direction makes with the normal an angle whose tangent is the ratio of the friction to the normal pressure. This angle has its greatest value when the friction is as great as possible.

The *angle of friction* is the angle between the resultant pressure and the normal when sliding is about to take place.

If this angle is denoted by ϕ, it follows from the definition that

$$\tan \phi = F/N = \mu.$$

That is, *the tangent of the angle of friction is equal to the coefficient of friction.*

131. Experimental Determination of Coefficient of Friction.— The value of the coefficient of friction may often be determined very simply. Thus, let a body be placed upon a horizontal plane surface and let a force be applied horizontally just great enough so that the body is on the point of sliding. This force is equal and opposite to the limiting friction. Its magnitude may be determined by a spring balance. The normal pressure is in this case equal to the weight of the body, which may also be determined by the spring balance. The ratio of these two forces is the coefficient of friction.

Another method of determining the coefficient of friction is as follows: Let a body be placed upon an inclined plane, and let no forces act upon it except its weight and the pressure of the supporting plane. This supporting pressure is exactly equal and opposite to the weight so long as the body is in equilibrium; the angle it makes with the normal is therefore equal to the inclination of the plane to the horizontal. Let this inclination be gradually increased until the body is on the point of sliding. The resultant pressure exerted by the plane now makes with the normal an angle equal to the angle of friction. Hence, if the inclination of the plane in this limiting position be measured, the coefficient of friction can be found from the relation

$$\mu = \tan \phi.$$

Many experiments have been made to determine the value of the coefficient of friction for different materials both with and without lubricants. The range of the results is indicated by the following values* of μ:

Wood on wood, dry	0.25 to 0.5
" " " soaped	0.2
Metals on oak, dry	2.5 to 0.6
" " " wet	0.24 to 0.26
" " " soaped	0.2
Metals on metals, dry	0.15 to 0.2
" " " wet	0.3
Smooth surfaces occasionally lubricated	0.07 to 0.08
" " thoroughly "	0.03 to 0.036

Examples.

1. A body of W lbs. mass is at rest upon a plane making angle θ with the horizontal. A cord attached to this body runs parallel to the plane, passes over a smooth pulley and sustains a weight of P

* See Encyclopædia Britannica, Vol. XV, p. 765.

lbs. Determine the magnitude and direction of the friction, the normal pressure, and the total pressure, exerted by the plane upon the body.

2. In Ex. 1, let $W = 50$ lbs., $P = 40$ lbs., $\theta = 32°$, and suppose the body is just about to slide up the plane. Determine the coefficient of friction. *Ans.* $\mu = 0.318$; $\phi = 17° \, 40'$.

3. In Ex. 1, if the coefficient of friction is 0.4 and θ is 30°, what must be the value of P (in terms of W) in order that the body may just slide up the plane? What value of P will just allow sliding down the plane? *Ans.* $0.846\, W$; $0.154\, W$.

4. A body of 30 lbs. mass, resting upon a plane inclined 45° to the horizontal, is pulled horizontally by a force P. If the coefficient of friction is 0.2, between what limits may the value of P vary and still permit the body to remain at rest? *Ans.* 20 lbs. and 45 lbs.

5. A straight bar of length l, whose center of gravity is distant a from the lower end, rests upon a horizontal surface and leans against a vertical wall. If the coefficient of friction between the floor and bar is μ, and the wall is perfectly smooth, what angle with the vertical may the bar make without sliding? *Ans.* $\tan^{-1}(\mu l/a)$.

6. If the conditions are as in Ex. 5, except that the vertical wall is rough, the coefficient of friction between the wall and the bar being μ', what is the position of incipient sliding?

7. What are the limiting positions of equilibrium for a heavy bead on a circular wire ring whose plane is vertical? (Express the result in terms of the coefficient of friction.)

8. What are the limiting positions of equilibrium of a uniform bar placed wholly within a spherical bowl? (Express in terms of the angle of friction.)

Ans. Let 2α = angle subtended at center by bar, θ = its angle with the horizontal in the limiting position; then $\tan\theta = \frac{1}{2}[\tan(\alpha + \phi) - \tan(\alpha - \phi)]$.

9. A bar AB whose mass is 6 lbs. and whose center of gravity is 4 ft. from the end A is supported in a horizontal position by a string attached at A and a peg 7 ft. from A. The coefficient of friction between the bar and the peg is 0.3. If the bar is about to slide in the direction AB, determine the direction of the string, the tension it sustains, and the supporting pressure exerted by the peg.

Ans. Tension = 2.77 lbs.; resistance of peg = 3.58 lbs.; angle between string and bar = 68° 12′.

10. In the preceding case, between what limits must the direction of the string lie in order that there may be equilibrium? Determine the corresponding limiting values of the supporting forces.

11. Solve the preceding problem in the following general case: Let W = weight of bar, a = distance of its center of gravity from A, b = distance of peg from A, μ = coefficient of friction.

132. Friction in Machines.— In order that a simple machine may produce its desired effect, its motion must be constrained, that is, guided in a definite manner. This is accomplished by means of bodies which touch the machine parts at certain points and exert constraining forces upon them. Besides the forces which produce the desired constraint, there are usually frictional forces which resist any tendency of the bodies to slide over each other at their surfaces of contact. In general friction is a *wasteful resistance;* but, as will be seen, it may in certain cases act *with* the effort and thus aid the useful object of the machine.

The effect of friction on the operation of a machine may be estimated by means of the laws of friction above stated. The method of applying them is illustrated in the following simple case.

133. Fixed Pulley with Friction.— If a pulley is mounted freely upon a cylindrical axle, the frictional force exerted by the axle upon the wheel always acts in such a way as to oppose whatever rotation the wheel has or tends to have by reason of the action of other forces.

Referring to Fig. 80, let a denote the radius of the pulley; r the radius of the axle; P the effort; W the load; R the resultant pressure exerted by the axle on the pulley; N the normal component of R; F the friction, or tangential component of R; μ the coefficient of friction; ϕ the angle of friction.

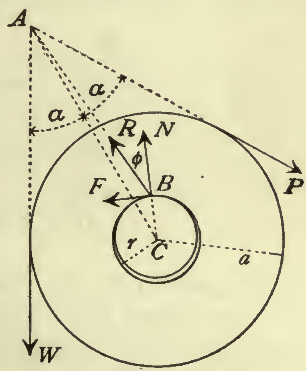


FIG. 80.

The cylindrical hole into which the axle fits is slightly larger than the axle, but they may be regarded as practically equal in size. If the wheel is on the point of moving *with* the effort, the equation of moments, taking origin at center of wheel, is

$$Pa - Wa - Fr = 0,$$

or

$$P = W + Fr/a, \quad . \quad . \quad . \quad (1)$$

which shows a mechanical disadvantage.

If the wheel is on the point of moving *against* the effort, the direction of F is reversed, and the equation is

$$P = W - Fr/a, \quad . \quad . \quad . \quad . \quad (2)$$

showing a mechanical advantage.

The value of F in case of incipient motion may be determined as follows:

In this limiting case,

$$F = N \tan \phi,$$

the total resistance R acting at an angle ϕ with the normal to the surfaces at the point of contact. But this total resistance must act in a line through the intersection of the lines of action of P and W. Let θ be the value of the angle CAB (Fig. 80); then

$$\frac{\sin \theta}{r} = \frac{\sin \phi}{a \operatorname{cosec} \alpha};$$

$$\therefore \quad \sin \theta = \frac{r}{a} \sin \phi \sin \alpha. \quad . \quad . \quad . \quad (3)$$

Since R, P and W are in equilibrium, we have

$$\frac{R}{\sin 2\alpha} = \frac{P}{\sin (\alpha + \theta)} = \frac{W}{\sin (\alpha - \theta)};$$

$$R = \frac{P \sin 2\alpha}{\sin (\alpha + \theta)} = \frac{W \sin 2\alpha}{\sin (\alpha - \theta)}; \quad . \quad . \quad (4)$$

$$F = R \sin \phi = \frac{W \sin 2\alpha \sin \phi}{\sin (\alpha - \theta)}. \quad . \quad . \quad . \quad (5)$$

From equation (5) the value of F in terms of W can be found after θ is determined from equation (3). Substituting the value of F in equation (1), the relation between P and W is determined.

The above discussion applies to the case in which there is incipient motion *with* the force P. If the opposite motion is about to occur, the form of the result is similar, but P and W must be interchanged.

EXAMPLES.

1. Let P and W (Fig. 80) be inclined to each other at an angle of 90°; radius of pulley = 6 ins.; radius of axle = ¾ in.; coefficient of friction = 0.2. Determine θ and the relation between P and W in case of incipient motion in each direction.

Ans. $\theta = 59'$; $P/W = 0.952$ or 1.050.

2. Prove that, if P and W are parallel, the relation between them is

$$P/W = (a + r \sin \phi)/(a - r \sin \phi)$$

or

$$P/W = (a - r \sin \phi)/(a + r \sin \phi),$$

according as there is incipient motion *with* P or *against* P. Show that the first of these equations is included as a special case in the above general solution.

3. With data as in Ex. 1, except that P and W are parallel, determine the relation between P and W for both cases of incipient motion. Determine also the value of F and of R in each case.

Ans. $P/W = 0.952$ or 1.050.

CHAPTER VIII.

EQUILIBRIUM OF FLEXIBLE CORDS.

134. Natural Bodies Not Perfectly Rigid.— A perfectly rigid body could not be deformed in any degree by the action of external forces. Actual bodies, however, undergo changes of form and size when forces are applied to them. In some cases the changes are so slight as to be of little importance. Thus, a steel beam, resting upon two supports, may bend only slightly under very heavy loads. For certain purposes this bending is unimportant, while for others it must be taken into account. The statics of non-rigid bodies is less simple than that of rigid bodies. Certain classes of problems may, however, be treated without great difficulty by means of the following general principle.

135. Non-Rigid Body in Equilibrium. —*The conditions of equilibrium for a rigid body apply to any body, every portion of which is in equilibrium.*

Suppose a body, acted upon by any external forces, to be deformed in any manner, reaching finally a condition of equilibrium. This condition being attained, the rigidity of the body is of no further consequence so far as its equilibrium is concerned. The external forces must therefore satisfy the same conditions as if the body were actually rigid.

136. Flexible and Inextensible Cord.— A *flexible cord* has been defined as one which may be bent by the application of external forces. A *perfectly flexible* cord would offer no resistance to bending.

The following discussions will relate to cords assumed to possess this ideal property of perfect flexibility. The results are of practical value, since many actual cords are flexible to a high degree. It will be assumed further that the cords are *inextensible.*

The forces acting upon a cord may be *concentrated* or *distributed.* (Art. 37). These two cases will be considered separately.

137. Geometrical Conditions of Equilibrium of Cord Acted Upon by Concentrated Forces.— Consider a perfectly flexible, weightless cord, suspended from two fixed points, and acted upon by

any concentrated forces. Thus, let X and Y (Fig. 81) be the fixed points of suspension, and let forces P_1, P_2, P_3 be applied at points L, M, N. Suppose the cord to have assumed a form of equilibrium, the segments XL, LM, MN, NY being straight. Let T_1, T_2, T_3, T_4 denote the tensions in the successive segments of the string, taken in order from X to Y. Then there is a definite relation between any applied force, as P_1, and the tensions in the adjacent portions of the string, as T_1 and T_2. By Art. 115 any portion of the string may be treated as a separate body, and the conditions of equilibrium may be applied to the forces acting upon it. For the portion of the string near L the forces are three, namely, P_1 and the forces due to the tensions in XL and LM,—that is, a force T_1 in the direction LX, and a force T_2 in the direction LM. Similarly, for the por-

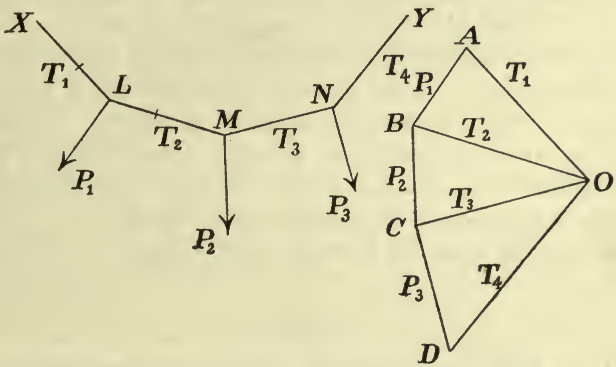


FIG. 81.

tion of the string near M we have three forces in equilibrium: the force P_2, a force T_2 in the direction ML, and a force T_3 in the direction MN. A similar analysis applies to the point N. If P_1, P_2 and P_3 are known, the tensions may be found, either geometrically or by writing the equations of equilibrium for each of the sets of forces named. The geometrical construction is shown in Fig. 81.

The line AB is drawn to represent P_1 in magnitude and direction; then BO and OA drawn parallel to LM and LX represent the forces T_2 and T_1. Draw BC to represent P_2 in magnitude and direction; then for the equilibrium of the portion of the cord near M, CO must 1epresent the force T_3. Similarly, if CD represents P_3 in magnitude and direction, DO must represent the force T_4. The

figure * thus shows at a glance the relations of magnitude and direction that must subsist among the forces P_1, P_2, P_3 and the tensions T_1, T_2, T_3, T_4.

Algebraically, the relations between these quantities may be expressed by writing two equations of equilibrium for each of the three systems of forces acting at L, M and N.

138. General Problem of Cord Acted Upon by Concentrated Forces.— If a cord fixed at two points is acted upon by concentrated forces, numerous problems may arise, depending upon what quantities are given and what are required. Thus, in the case represented in Fig. 81, the quantities involved are the applied forces P_1, P_2 and P_3; the tensions T_1, T_2, T_3, T_4; and the coördinates which give the positions of the points X, L, M, N, Y. (Instead of these coördinates, we may use the lengths and directions of the segments XL, LM, etc.) If enough of these quantities are known the others may be determined. Many of the problems which may arise do not admit of simple treatment. If the problem is determinate, as many equations may be written as there are unknown quantities. Part of these are the equations of equilibrium, written for each point at which a force is applied in the manner above indicated; the others must be written from the geometrical relations of the figure.

The following examples illustrate simple cases.

Examples.

1. Let a single vertical load of P lbs. be suspended from a cord at distances l and m from its ends; and let the ends X and Y be fixed in such positions that the distance XY is a and the line XY makes with the horizontal an angle θ. Determine the tensions in the cord.

2. In the preceding example let $l = 10$ ins., $m = 8$ ins., $a = 12$ ins., $\theta = 30°$. *Ans.* $0.0744\ P$ and $0.988\ P$.

3. Let the ends of the cord be fixed at points in the same horizontal line at a distance apart equal to a, and let vertical loads P and Q be suspended at points dividing the length of the cord into segments l, m and n. Determine the relation between P and Q so that

---

* The construction here described is of importance in Graphical Statics. The polygon formed by the successive segments of the cord is called a *funicular polygon*. (Sometimes called also *equilibrium polygon* and *string polygon*.)

the middle segment may be horizontal. Prove that, if the middle segment is horizontal, the following relation must be satisfied: $(P-Q)/(P+Q)=(n^2-l^2)/(a-m)^2$.

4. If, in the preceding example, $a=12$ ins., $l=4$ ins., $m=6$ ins., $n=6$ ins., prove that $P=3.5Q$. Determine the tensions, assuming $Q=10$ lbs., $P=35$ lbs.

Ans. Tension in horizontal cord $=12.37$ lbs.

139. Equations of Equilibrium for Any Part of Cord Carrying Distributed Vertical Load.— Let it be required to determine the form of the curve assumed by a cord suspended from two fixed points and sustaining a vertical load distributed along the cord in any manner.

Let the coördinates of any point of the curve (Fig. 82) be x and y, the axis of x being horizontal and that of y vertical, and the origin being any point in the plane of the curve. Let ϕ denote the angle between the tangent to the curve at any point and the axis of x, and T the tension in the cord at that point. Also let w be the load sustained by the cord per unit length (made up of its own weight and any other applied load). In general w varies continuously along the cord.

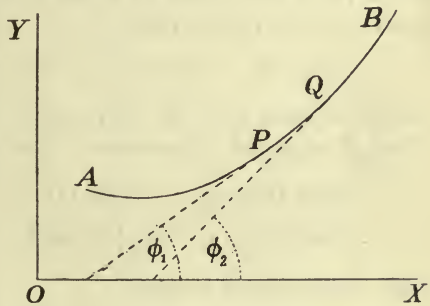


FIG. 82.

Let P and Q be any two points of the cord, and let the values of x, y, T and ϕ at P be denoted by x_1, y_1, T_1, ϕ_1; while at Q the values are x_2, y_2, T_2, ϕ_2. Let W represent the total load applied to the portion PQ. Applying the conditions of equilibrium to PQ, and resolving forces horizontally,

$$T_2 \cos \phi_2 - T_1 \cos \phi_1 = 0; \qquad (1)$$

resolving vertically,

$$T_2 \sin \phi_2 - T_1 \sin \phi_1 - W = 0. \qquad (2)$$

A moment equation might also be written, but is not needed in the present discussion.

Equation (1) expresses the fact that the horizontal component of

the tension has the same value at all points of the cord. Let this value be represented by H, so that

$$T_1 \cos \phi_1 = T_2 \cos \phi_2 = T \cos \phi = H = \text{constant.} \quad . \quad (3)$$

Substituting in equation (2) the values $T_1 = H / \cos \phi_1$, $T_2 = H / \cos \phi_2$, that equation becomes

$$\tan \phi_2 - \tan \phi_1 = W / H. \quad . \quad . \quad . \quad (4)$$

Equations (3) and (4) may be used instead of (1) and (2).

If the values of T and ϕ are known at any one point of the cord, H becomes known. If also W is known for every portion of the cord, equation (4) serves to compute the slope of the curve at any point. A differential equation may, however, be deduced which is more useful.

140. Differential Equation of Curve of Loaded Cord.—Let s denote the length of the curve measured from some fixed point up to the point (x, y), and let s_1 and s_2 be the values of s at P and Q. From equation (4) we have

$$(\tan \phi_2 - \tan \phi_1) / (s_2 - s_1) = W / H (s_2 - s_1).$$

Let the limiting value of each member of this equation be found as P and Q approach coincidence. Evidently,

$$\text{limit } [(\tan \phi_2 - \tan \phi_1) / (s_2 - s_1)] = d(\tan \phi) / ds;$$

$$\text{limit } [W / (s_2 - s_1)] = w.$$

Hence the equation becomes

$$d(\tan \phi) / ds = w / H. \quad . \quad . \quad . \quad (5)$$

To apply this equation to any particular case, the distribution of loading must be known, so that w can be expressed in terms of the coördinates of the curve. Two important cases will be considered.

141. The Common Catenary.— The curve assumed by a cord whose weight per unit length is uniform, is called the *common catenary.*

In this case w is constant; let $H / w = c$, so that the constant c denotes the length of cord whose weight is equal to the horizontal tension H. Writing p for $\tan \phi$, equation (5) becomes

$$dp / ds = 1 / c. \quad . \quad . \quad . \quad . \quad (6)$$

Since $dp/ds = (dp/dx)(dx/ds) = (dp/dx) \cos \phi$, and $\cos \phi = 1/\sqrt{1 + p^2}$, equation (6) may be written

$$dp/\sqrt{1 + p^2} = dx/c. \qquad (7)$$

Integrating,

$$\log \left(p + \sqrt{1 + p^2}\right) = x/c + K.$$

To determine the constant K, suppose the tangent of the curve at some point to be horizontal, and let the origin of coördinates lie in a vertical line through that point, so that when x equals zero, p equals zero. Then K equals zero, and the equation may be written

$$p + \sqrt{1 + p^2} = e^{x/c}. \qquad (8)$$

Solving for p,

$$p = dy/dx = \tfrac{1}{2}(e^{x/c} - e^{-x/c}). \qquad (9)$$

Integrating,

$$y = \tfrac{1}{2}c(e^{x/c} + e^{-x/c}) + K'. \qquad (10)$$

If the origin of coördinates is taken at the lowest point of the curve, so that $y = 0$ when $x = 0$, K' is equal to $-c$. If the origin is at a distance c below the lowest point of the curve, $K' = 0$. Adopting the latter supposition, the equation becomes

$$y = \tfrac{1}{2}c(e^{x/c} + e^{-x/c}). \qquad (11)$$

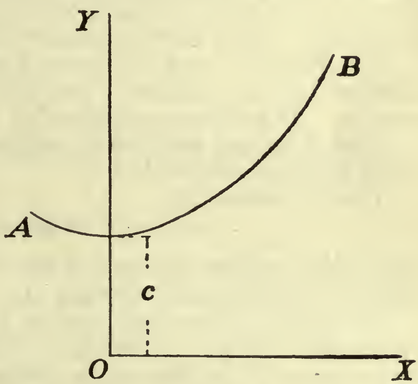


FIG. 83.

In the solution of problems relating to the catenary, use is made of certain relations between the following quantities: the length of the cord; the tension at any point; the tension at the lowest point; the inclination of the tangent at any point; the coördinates of any point. The following equations may be deduced:

(*a*) From equation (6), by integration,

$$s = cp = c \tan \phi; \qquad (12)$$

the constant of integration being zero if s is reckoned from the point at which $\phi = 0$, that is, from the lowest point of the curve.

(*b*) Eliminating x between equations (9) and (11), there results

$$y = c\sqrt{1 + p^2} = c \sec \phi. \quad . \quad . \quad . \quad (13)$$

(*c*) Eliminating p between (12) and (13),

$$y^2 - s^2 = c^2. \quad . \quad . \quad . \quad . \quad (14)$$

The use of these equations will now be illustrated.

EXAMPLES.

1. Let a cord be suspended from two points at a given distance apart in the same horizontal line, the load per unit length (w) being known, and the tension at the lowest point (H) being assigned. Required (1) the length of the curve, (2) the position of the lowest point, (3) the slope at points of suspension, (4) the tension at points of suspension.

Since w and H are given, c is known, its value being H/w. Also, the value of x at a point of suspension is known (being half the distance between the two given points); hence from (11) the value of y at a point of suspension can be computed. Subtracting c from this value of y gives the depth of the lowest point below the points of suspension; which answers question (2).

The length of half the curve may be found by substituting in equation (14) the value of c and the value of y at the point of suspension. This answers question (1).

From (13) may be found the value of ϕ at the point of suspension, which answers question (3).

From equation (3),

$$T = H \sec \phi,$$

from which may be determined the value of T at the points of suspension, thus answering question (4).

In solving a numerical case of this example, the value of the exponential terms in equation (11) must be found by logarithms. The value of e, the base of the system of hyperbolic logarithms, is 2.7182818 and its common logarithm is 0.4342945.

2. Let the points of suspension be 100 ft. apart in the same horizontal plane, let the load per foot be 50 lbs., and let it be required that the tension at the lowest point shall be 1000 lbs.

From the given data,

$$c = H/w = 1000/50 = 20 \text{ ft.}$$

The coördinates of one of the points of suspension are

$$x = 50 \text{ ft.}, \quad y = 10\,(e^{2.5} + e^{-2.5}) = 122.6 \text{ ft.}$$

The lowest point of the curve is therefore 102.6 ft. lower than the points of suspension.

For the half length of the curve,

$$s^2 = y^2 - c^2 = (122.6)^2 - (20)^2;$$
$$s = 121.0 \text{ ft.}$$

For the value of ϕ at the point of suspension we have

$$\cos\phi = c/y = 20/122.6 = 0.1631\,; \quad \phi = 80^\circ\ 37'.$$

The value of T at the end is

$$T = H \sec\phi = 6,130 \text{ lbs.}$$

3. A cord of known length is suspended from two given points in the same horizontal plane; determine the position of the lowest point of the curve, and the tension at any point.

In this case the value of x at the end of the cord is known, but the corresponding value of y is unknown, as is also the value of c. To solve the problem, y and c must be determined so as to satisfy equations (11) and (14). This is best accomplished by trial when numerical data are given.

142. Load Distributed Uniformly Along the Horizontal.—Let w' denote the load per unit horizontal distance. Since the load on a length ds of the curve falls upon a horizontal distance dx,

$$w\,ds = w'dx,$$

or

$$w' = w(ds/dx) = w \sec\phi. \qquad (15)$$

Consider now the case in which w' is constant.

Writing p for $\tan\phi$, and substituting $w'\cos\phi$ for w, equation (5) (Art. 140) may be written

$$dp/ds = (w'\cos\phi)/H = (\cos\phi)/c', \qquad (16)$$

in which c' is written for H/w', and is a constant denoting the horizontal distance upon which the load is equal to the horizontal tension H. Again, since $dp/ds = (dp/dx)(dx/ds) = (dp/dx)\cos\phi$, the equation becomes

$$dp/dx = d^2y/dx^2 = 1/c'. \qquad (17)$$

Integrating, taking the axis of y through the point where the curve is horizontal, so that when $x = 0$, $dy/dx = 0$, there results

$$p = dy/dx = x/c'. \qquad (18)$$

Integrating again, taking the origin on the curve, so that $y = 0$ when $x = 0$,

$$y = x^2/2c'. \qquad (19)$$

This represents a parabola with vertex at the origin of coördinates and axis vertical.

The *length of the curve*, measured from the origin to any point, may be determined as follows:

Since $\cos\phi = 1/\sqrt{1+p^2}$, equation (16) may be written

$$ds = c'\sqrt{1+p^2}\cdot dp,$$

the integral equation of which is

$$s = \tfrac{1}{2}c'[p\sqrt{1+p^2} + \log(p + \sqrt{1+p^2})], \quad . \quad (20)$$

the constant of integration being zero if $s = 0$ when $p = 0$. If it is desired to express s in terms of x, p may be eliminated by means of equation (18).

EXAMPLES.

1. A cord carrying a known load with uniform horizontal distribution is suspended from two points on the same level at a given distance apart. The lowest point of the cord is at a known distance below the points of suspension. To determine (*a*) any number of points of the curve; (*b*) the tension at any point; (*c*) the length of the cord.

(*a*) Since the values of x and y at the points of suspension are known, c' can be determined from equation (19); after which the same equation serves to determine the coördinates of any number of points of the curve.

(*b*) The value of w' being known, H can be computed from the relation $H = c'w'$. The slope at any point may be found from the equation

$$\tan\phi = p = dy/dx = x/c',$$

and the tension at any point by the relation

$$T = H\sec\phi = H\sqrt{1+p^2}.$$

(*c*) The length of the cord from the lowest point up to any given point may be found from equation (20), using the above value of p.

(2) A cord carrying 40 lbs. per horizontal foot is suspended at two points on the same level 40 ft. apart, so that the lowest point of the cord is 10 ft. below the point of suspension. Required the tension at the lowest point; the tension at a point of suspension; and the length of the cord.

143. Approximate Solution of Problem of Loaded Cord.— If a cord is suspended from two points on the same level whose distance apart differs little from the length of the cord, the relation between the load, the sag,* the distance between points of sus-

* By the sag is meant the vertical distance of the lowest point of the cord below the level of the points of support.

pension and the length may be determined approximately as follows:

If the weight per unit length is uniform, it may be assumed, with small error, that the load is uniformly distributed along the horizontal, and that $w' = w$. On these assumptions the curve is a parabola and equations (19) and (20) of Art. 142 are applicable. An approximate value of s may be obtained in simple form.

If the value of s given by equation (20) be developed in powers of p, the first two terms of the result are

$$s = c'(p + p^3/6).$$

This result, expressed in terms of x, is

$$s = c'(x/c' + x^3/6c'^3) = x\,(1 + x^2/6c'^2).$$

EXAMPLES.

1. A wire weighing 0.1 lb. per yard of length is suspended between two points 100 ft. apart so that the sag at the middle is 2 ft. Determine the length and the greatest tension.

The equation of the curve being $y = x^2/2c'$, the value of c', found by substituting in this equation the coördinates of any point of the curve, is

$$c' = x^2/2y = 2500/4 = 625 \text{ ft.}$$

The tension at the lowest point is

$$H = c'w' = 625/30 = 20.8 \text{ lbs.}$$

The value of p at the support is

$$p = x/c' = 50/625 = 0.08,$$

and the value of the tension at the point of suspension is therefore

$$T = H \sec \phi = 20.8\sqrt{1 + 0.0064} = 20.9 \text{ lbs.}$$

The half length of the curve is

$$s = c'(p + p^3/6) = 625\,(0.08 + 0.000512/6) = 50.053 \text{ ft.}$$

2. Solve Ex. 1, assuming the sag equal to (*a*) 1 ft. and (*b*) 3 ft.

144. Suspension Bridge.—The cables of a suspension bridge usually hang in parabolic curves. If the total load carried by such a cable has a uniform horizontal distribution, there is no tendency of the cables to depart from the parabolic form. If, however, this distribution of loading is departed from, there is a tendency to change the form of the curve assumed by the cable. To counteract this tendency to distortion is the office of the "stiffening truss."

For a complete discussion of the theory of suspension bridges, works on the theory of bridges must be consulted.

CHAPTER IX.

CENTROIDS.

§ 1. *Centroid of a System of Parallel Forces.*

145. Forces with Fixed Points of Application.— In the discussion of systems of forces applied to rigid bodies, it has been assumed that a force may be applied at any point in its line of action, since its effect upon the motion (or tendency to motion) of the body is the same for all such points of application. In certain cases, however, forces are applied at definite points which remain fixed in the body, whatever displacement it may undergo. Thus, the force of gravity acting upon a body is the resultant of forces applied to every particle of the body and these points of application remain fixed in the body, however it may be turned from its original position.

In the following discussion of systems of forces with points of application fixed in the body, it will be assumed that the forces are parallel, and that they remain constant in magnitude and direction, however the body may be displaced.

It obviously amounts to the same thing, so far as the relations of the forces are concerned, whether their direction remains fixed while the body turns, or whether their direction changes while the body remains fixed.

146. Resultant of Two Parallel Forces with Fixed Points of Application.— Let two parallel forces be applied at fixed points, A and B. Their resultant is equal to their algebraic sum, and its line of action divides the line AB into segments inversely proportional to the given forces (Art. 86). Let C (Fig. 84) be the point in which the line of action of the resultant intersects AB. If the two given forces remain parallel while their direction changes in any manner, their points of application still being A and B, the line of action of the resultant will continue to pass through the same point C; hence C may be taken as the fixed point of application of the resultant.

147. Resultant of Any Number of Parallel Forces with Fixed Points of Application.— If, in addition to the two parallel forces applied at A and B, there is a third force parallel to them applied at

a point D, the resultant of the three may be found by combining the third with the resultant of the first two. Since the resultant of the first two has a fixed point of application for all directions in which the forces may act, and since a point may also be found which may be regarded as the fixed point of application of the resultant of this force and the third of the given forces, it follows that the line of action of the resultant of the three given forces will always pass through a certain point (as E, Fig. 84), which may be taken as its fixed point of application. Since this process may be continued to include any number of forces, it is seen that the resultant of any number of parallel forces with fixed points of application acts in a line which always passes through a certain fixed point, whatever the direction of the forces.

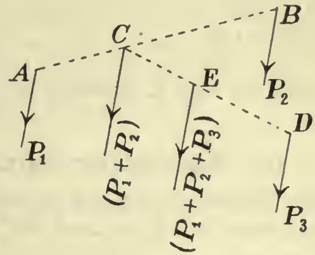


FIG. 84.

148. Centroid.— The point of application of the resultant of any system of parallel forces with fixed points of application is called the *centroid* of the system. Methods of determining the centroid will now be considered.

149. Determination of Centroid of Coplanar Forces.— Let P_1, P_2, . . . represent the given forces, and let the coördinates of their points of application with reference to a pair of rectangular axes be $(x_1, y_1), (x_2, y_2)$, ... Let R denote the resultant and $\bar{x}, \bar{y}$ the coördinates of its point of application, that is, the coördinates of the centroid of the system. Assume the forces to act first parallel to the axis of y and then parallel to the axis of x. The magnitude of the resultant is the same in both cases, and in each case it acts in a line which contains the centroid. Since the moment of the resultant of any system of forces is equal to the algebraic sum of the moments of the several forces, we may write, for the case in which the forces act parallel to the axis of y, the equation

$$R\bar{x} = P_1x_1 + P_2x_2 + \quad . \quad . \quad . \quad ;$$

and for the case in which they act parallel to the x-axis,

$$R\bar{y} = P_1y_1 + P_2y_2 + \quad . \quad . \quad . \quad .$$

From these, since R is the algebraic sum of the given forces, the coördinates of the centroid are found to be

$$\bar{x} = (P_1x_1 + P_2x_2 + \ .\ .\ .\ )/(P_1 + P_2 + \ .\ .\ .\ ) = \Sigma Px/\Sigma P;$$
$$\bar{y} = (P_1y_1 + P_2y_2 + \ .\ .\ .\ )/(P_1 + P_2 + \ .\ .\ .\ ) = \Sigma Py/\Sigma P.$$

Here the sign Σ denotes the summation of a series of terms of similar form.

150. **Non-Coplanar Parallel Forces.**—It will be well to extend the discussion to cases in which the forces, though parallel, are not restricted to a plane and the points of application have any positions in space.

The reasoning of Arts. 146 and 147, showing that every system of parallel forces with fixed points of application has a centroid, applies to this general case. To determine the position of the centroid, the following method may be employed.

(*a*) *Coplanar points of application.*—If the points of application are coplanar, while the direction of the forces is unrestricted, the centroid may be determined by assuming the forces to act in the plane of the points of application. If the axes of x and y are chosen in this plane, the coördinates of the centroid are given by the formulas deduced for coplanar forces.

(*b*) *Non-coplanar points of application.*—Let z_1, z_2, etc., denote the ordinates of the points of application of the several forces measured from any assumed reference plane. Any *two* of the given points of application lie in a plane perpendicular to the reference plane; hence the formula deduced for the case of coplanar points of application applies to any two forces. Or, if z' is the ordinate of the centroid of P_1 and P_2,

$$z' = (P_1z_1 + P_2z_2)/(P_1 + P_2).$$

Similar reasoning applies to the two forces $(P_1 + P_2)$ and P_3; if z'' denotes the ordinate of the centroid of these forces,

$$\begin{aligned} z'' &= [(P_1 + P_2)\,z' + P_3z_3]/[(P_1 + P_2) + P_3] \\ &= (P_1z_1 + P_2z_2 + P_3z_3)/(P_1 + P_2 + P_3). \end{aligned}$$

By extending this reasoning, any number of forces may be included, with the result that the ordinate of the centroid is

$$\bar{z} = (P_1z_1 + P_2z_2 + \ .\ .\ .\ )/(P_1 + P_2 + \ .\ .\ .\ ) = \Sigma Pz/\Sigma P.$$

Since the plane of reference may be any plane whatever, an indefinite number of equations of the above form may be written. Three equations, however, suffice for the determination of the centroid. If three rectangular coördinate planes are selected, and the points of application of the forces are (x_1, y_1, z_1), (x_2, y_2, z_2), etc., the coördinates of the centroid are given by the three equations

$$\bar{x} = \Sigma Px/\Sigma P; \quad \bar{y} = \Sigma Py/\Sigma P; \quad \bar{z} = \Sigma Pz/\Sigma P.$$

§ 2. *Centroids of Masses, Volumes, Areas and Lines.—General Method.*

151. Center of Mass Defined.— If parallel forces be conceived to act in the same direction upon all portions of a body, such that the resultant forces acting upon any two portions, however small, are proportional to their masses, the centroid of this system of forces is called the *center of mass* of the body.

The forces of gravity acting upon all portions of a body form a system which is practically such as described in this definition. The centroid of the forces of gravity is called the *center of gravity* of the body. The terms center of mass and center of gravity are often used interchangeably, although it is to be remembered that the forces of gravity do not *exactly* satisfy the conditions assumed in the definition of center of mass.

The determination of the mass-center of a body or of a system of bodies is a special case of the determination of the centroid of a system of parallel forces. The word *centroid* is, in fact, often used to designate the center of mass.

152. General Method of Determining Center of Mass.— Let the given body be divided into any number of parts whose masses are $m_1, m_2, \ldots$, and let the coördinates of their respective centers of mass be (x_1, y_1, z_1), (x_2, y_2, z_2), ... Assuming forces such as described in the definition of mass-center (Art. 151) to act upon all portions of the body, the problem is to find the point of application of the resultant of these forces. Now, from the definition of mass-center it follows that the forces acting upon all parts of m_1 may be replaced by their resultant, taken as applied at the mass-center of m_1; and similarly with each of the other portions. Hence, if $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coördinates of the mass-center of the whole body, there may

be written three equations similar to those which give the coördinates of the centroid of any system of parallel forces (Art. 150); the force-magnitudes being replaced by the masses m_1, m_2, ... The equations are, therefore,

$$\bar{x} = \Sigma mx / \Sigma m; \quad \bar{y} = \Sigma my / \Sigma m; \quad \bar{z} = \Sigma mz / \Sigma m.$$

To apply these formulas, it is necessary to know the center of mass of each of the separate masses. If the given body can be subdivided into finite parts whose masses and mass-centers are known, the application of the formulas is simple. If such subdivision is impossible, resort must be had to the integral calculus, if an exact determination is required.

The above formulas are directly applicable if it is desired to determine the center of mass of any number of bodies taken together, their several masses and mass-centers being known.

Examples.

1. Find the center of mass of a system of four bodies whose masses are 10 lbs., 50 lbs., 12 lbs., 40 lbs; their several mass-centers being at the points whose rectangular coördinates are (10, 4, —3), (12, —3, 2), (4, —5, —6), (—8, 3, 6).

2. Show that the mass-center of three particles of equal mass placed at the vertices of a triangle lies at the intersection of the medians.

3. Show that the mass-center of four particles of equal mass placed at the vertices of a triangular pyramid is at a distance from the base equal to one-fourth the altitude.

153. Density.— A body is said to be *homogeneous* if the masses of any two portions, however small, are proportional to their volumes.

The *density* of a homogeneous body is a quantity proportional directly to the mass and inversely to the volume. If the unit density is that of a body of which unit volume contains unit mass, the density of any homogeneous body is equal to the *mass of unit volume* of the body.

If ρ denotes the density, V the volume and M the mass,

$$\rho = M / V, \quad \text{or} \quad M = \rho V,$$

for a homogeneous body.

Dimensions of unit density.—The unit density above described is a derived unit (Art. 13), depending upon the units of mass and

length. This dependence is expressed by the dimensional equation (Art. 15)

$$(\text{unit density}) = (\text{unit mass})/(\text{unit volume}) = \mathbf{M}/\mathbf{L}^3.$$

If the pound is taken as the unit mass and the foot as the unit length, the unit density is that of a body of which each cubic foot of volume contains one pound mass. The density of any body is then expressed as a certain number of pounds per cubic foot. Other units of mass and length give other units of density,—for example a gram per cubic centimeter, or a kilogram per cubic meter. The methods of determining center of mass are independent of the particular unit of density.

Average density.—If a body is not homogeneous, its average density may be defined as the density of a homogeneous body whose total mass and total volume are equal to those of the given body. The average density is thus equal to the whole mass divided by the whole volume.

Actual density at a point.—In case of a heterogeneous body, the average densities of different portions are unequal. The conception of actual density *at a point* may be reached as follows:

Let ΔM be the mass and ΔV the volume of any portion containing the given point; its average density ρ' is then

$$\rho' = \Delta M/\Delta V.$$

If ΔV become smaller, approaching zero as a limit, but always containing the given point, ΔM also generally approaches zero, and ρ' approaches a definite limit ρ. This limiting value of ρ' is the density at the point considered. That is, the required density is given by the equation

$$\rho = dM/dV.$$

154. **Mass-Center of Homogeneous Body or System of Bodies.**— For a homogeneous body, or for a system of such bodies of equal densities, volumes may be substituted for masses in the formulas for the coördinates of the mass-center. For if ρ is the density of each body, and v_1, v_2, . . . are the volumes corresponding to masses m_1, m_2, . . . , we have

$$m_1 = \rho v_1, \quad m_2 = \rho v_2, \quad . \; . \; . \; ;$$

and substituting these values in the formulas for $\bar{x}$, $\bar{y}$, and $\bar{z}$, the factor ρ cancels, leaving

$$\bar{x} = \Sigma vx/\Sigma v; \quad \bar{y} = \Sigma vy/\Sigma v; \quad \bar{z} = \Sigma vz/\Sigma v.$$

155. Centroids of Volumes, Surfaces and Lines.— *Volumes.*— If parallel forces be conceived to be applied to all portions of a body, the resultant forces acting upon any two portions, however small, being proportional to their volumes, the centroid of this system of forces is called the *center of volume* (or *volume-centroid*) of the body.

The centroid of a given volume evidently coincides with the mass-center of a homogeneous body occupying the volume, and may be found by the formulas already given.

Surfaces.—The centroid (or center of area) of any surface is the centroid of a system of parallel forces conceived to be applied to all portions of the surface, the resultant forces acting upon any two portions, however small, being proportional to their areas. This definition applies to curved surfaces as well as to plane areas. In case of a plane surface the centroid lies in the plane; but the centroid of a curved surface does not in general lie in the surface.

The formulas for the coördinates of the centroid of a homogeneous body or of a volume may be applied in finding the centroid of an area, if v_1, v_2, . . . represent partial areas and (x_1, y_1, z_1), (x_2, y_2, z_2) . . . the coördinates of their centroids.

Lines.—The centroid (or center of length) of any line is the centroid of a system of parallel forces conceived to be applied to every portion of the line, the resultant forces acting upon any two portions, however small, being proportional to their lengths.

If the line is divided into parts whose lengths and centroids are known, the centroid of the whole line is found by formulas identical in form with those used for finding the coördinates of the centroid of a volume or area; but v_1, v_2, . . . must represent lengths.

156. Moment with Respect to a Plane.— The *moment of a mass* with respect to a plane is defined as the product of the mass into the distance of its mass-center from the plane.

The *moment of a volume* with respect to a plane is the product of the volume into the distance of its centroid from the plane.

The *moment of a surface* (or *of a line*) with respect to a plane is the product of its area (or length) into the distance of its centroid from the plane.

The formula for the x-coördinate of the centroid of any mass (Art. 152) may be written

$$(m_1 + m_2 + \ldots)\bar{x} = m_1 x_1 + m_2 x_2 + \ldots .$$

Since the plane from which x is measured may be any plane, the equation expresses the proposition that

The moment of any mass with respect to a plane is equal to the sum of the moments of any parts into which it may be divided.

The same proposition obviously holds for a volume, a surface or a line.

As a special case, if the given plane contains the centroid of the body, surface or line, the sum of the moments of the parts is zero.

157. **Symmetry.**— If a volume, a surface or a line has a plane of symmetry, the centroid lies in this plane. For the volume, surface or line can be divided into pairs of equal elements such that the centroid of each pair lies in the plane of symmetry.

Similarly, if there is an axis of symmetry it contains the centroid; and if there is a center of symmetry it coincides with the centroid.

A similar proposition is true for a mass if homogeneous.

From the principles of symmetry the centroids of many geometrical figures may be located, either completely or partly, by inspection. Thus:

The centroid of a straight line is at its middle point.

The centroid of a circular arc lies upon the line bisecting the angle subtended at the center.

The centroid of a rectangular area lies upon the line bisecting two opposite sides, and is therefore at the intersection of the two lines so drawn.

The centroid of the area of an ellipse is at its center of figure.

The centroid of an ellipsoid is at its center.

158. **Centroids of Plane Figures and of Volumes.**—*Parallelogram.*— Let $ABCD$ (Fig. 85) be a parallelogram. Since the relation of the area to AB and its relation to the opposite side CD are exactly similar, the centroid is equally distant from these two lines. For a like reason it is equally distant from AD and BC. It is therefore at the intersection of the bisectors LM, HK. This point of intersection is, in fact, the center of symmetry of the parallelogram. It is also the point of intersection of the diagonals.

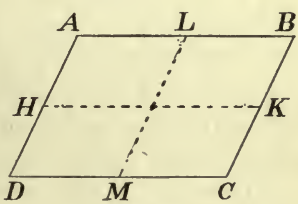


FIG. 85.

Triangle.— In the triangle ABC (Fig. 86) draw AX bisecting the side BC, and draw lines such as $B'C'$, parallel to BC and limited by AB and AC. These lines are all bisected by AX. Draw $B'b$ and $C'c$ parallel to AX, thus constructing a parallelogram $B'C'cb$. The centroid of every such paralelogram lies on AX, hence the centroid of the figure made up of all of them lies on AX. If the number of the parallelograms be increased indefinitely, the width of each approaching zero, this figure approaches coincidence with the triangle; hence the centroid of the triangle lies upon AX. Similar reasoning shows that it lies on the line drawn from C bisecting AB, and on the line drawn from B to the middle point of AC. Hence it is at the common point of intersection of these three lines.

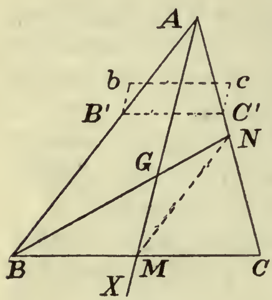


FIG. 86.

Let M (Fig. 86) be the middle point of BC, N the middle point of AC, and G the point of intersection of AM and BN. By comparison of the similar triangles ABC and NMC, it is seen that NM is equal to half of AB. Comparing the similar triangles GAB and GMN, it is seen that GM is half of AG and GN half of BG. Hence GM is one-third of AM, and GN one-third of BN.

Any plane polygon, regular or irregular, may be subdivided into triangles whose areas and centroids may be computed; so that the centroid of the whole area can be determined by the general formulas of Art. 154.

Right prism.—A right prism may be generated by the motion of a plane area perpendicular to itself. The centroid of the area will describe a line which must contain the centroid of the prism. For, any elementary areas will generate volumes proportional to the areas; and if moments be taken with respect to any plane containing the line generated by the centroid of the generating area, the moments of the elementary volumes will be proportional to the moments of their generating areas. But the moment of the whole area is zero for such a plane; hence the moment of the whole volume is also zero, and its centroid therefore lies in the plane. The centroid is obviously equidistant from the two bases of the prism.

Oblique prism.—An oblique prism may be generated by the rectilinear motion of a plane area in a direction not perpendicular to itself. The reasoning given for the case of a right prism may be applied to this case, with a like result.

Triangular pyramid.—The centroid of the volume of a triangular pyramid lies in the line joining the vertex with the centroid of the base. To prove this, let A be the vertex, BCD the base, and M the centroid of the base (Fig. 87). Let $B'C'D'$ be a plane section* parallel to BCD, M' being its centroid; it may be proved without difficulty that M' lies on the straight line AM. Let $B''C''D''$ be another plane section parallel to the base, and let a prism be generated by the translation of $B'C'D'$ parallel to AM until it falls into the plane $B''C''D''$. The centroid of this prism lies in AM. Let any number of plane sections parallel to BCD be taken, and let each be the base of a prism generated in the same way as the one described. Since the centroid of each prism lies upon AM, so also does the centroid of their combined volume. If the number of plane sections be increased indefinitely, their distance apart approaching zero as a limit, the volume of each elementary prism approaches zero, but their combined volume approaches that of the pyramid; hence the centroid of the pyramid lies upon AM.

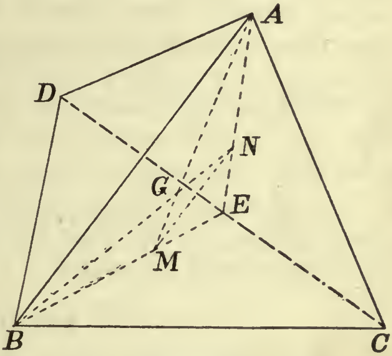


FIG. 87.

To determine the position of the centroid, let N (Fig. 87) be the centroid of the face ACD; then the centroid of the pyramid must lie on BN, and must therefore be the point of intersection of AM and BN. If E is the middle point of CD, M lies upon EB, and $EM = EB/3$; also, N lies upon EA, and $EN = EA/3$. The triangles EAB and ENM are therefore similar, and $NM =$

* The sections $B'C'D'$ and $B''C''D''$ are not shown in Fig. 87; but it is to be understood that like letters denote corresponding vertices of the three triangles BCD, $B'C'D'$, $B''C''D''$.

$AB/3$. Again, comparing the similar triangles GAB, GMN, it is seen that

$$GM = AG/3 = AM/4;$$
$$GN = BG/3 = BN/4.$$

Therefore, *the distance of the centroid from the base is equal to one-fourth the altitude.*

Any pyramid or cone.—The above proof that the centroid lies upon the line drawn from the vertex to the centroid of the base applies to a pyramid whose base is any polygon. Moreover, if the base be divided into triangles, the pyramid can be divided into triangular pyramids whose centroids all lie in a plane parallel to the base and at a distance from it equal to one-fourth the altitude; hence the centroid of the given pyramid also lies in that plane.

If the base is bounded by any curve, this may be regarded as the limit of an inscribed polygon the number of whose sides is increased indefinitely. Hence the following proposition may be stated:

The centroid of any pyramid or cone lies in the line joining the vertex with the centroid of the base, and at a distance from the base cqual to one-fourth the altitude.

Examples.

1. Find the centroid of the area of half of a regular hexagon.

2. Determine the centroid of an area composed of a square, and an equilateral triangle one of whose sides coincides with a side of the square.

3. Determine the centroid of a volume composed of a cube, and a right pyramid whose base coincides with a face of the cube and whose altitude is equal to the length of an edge of the cube.

4. Find the centroid of a volume made up of a right circular cylinder of given base and altitude, and a cone of any altitude whose base coincides with a base of the cylinder.

5. Prove that the centroid of the area of a triangle coincides with the mass-center of a system of three particles of equal mass situated at the vertices of the triangle.

6. Determine the centroid of the volume of a frustum of a cone or pyramid.

7. Prove that the centroid of the volume of a tetrahedron coincides with the mass-center of a system of four particles of equal mass situated at the vertices of the tetrahedron.

§ 3. *Determination of Centroids by Integration.*

159. General Formulas.—In many cases it is not possible to subdivide the body, surface or line whose centroid is required into finite portions whose centroids are known. In such cases, if an exact result is desired, resort must be had to integration.

Starting with the general formula

$$\bar{x} = (m_1x_1 + m_2x_2 + \ldots)/(m_1 + m_2 + \ldots),$$

let the number of parts into which the body is divided be increased, while their volumes and masses become smaller, approaching zero. Each term in the sum

$$m_1x_1 + m_2x_2 + \ldots$$

approaches zero, while the sum in general approaches a finite limit, which has the value

$$\text{limit } [m_1x_1 + m_2x_2 + \ldots] = \int x\,dM,$$

the limits of the integration being so taken as to include the whole body. The denominator in the expression for $\bar{x}$ is equal to M (the whole mass of the body), and may be written in the form $\int dM$. The values of $\bar{x}$, $\bar{y}$ and $\bar{z}$ may therefore be written

$$\bar{x} = \int x\,dM/\int dM; \quad \bar{y} = \int y\,dM/\int dM; \quad \bar{z} = \int z\,dM/\int dM. \quad (1)$$

If ρ denotes the density at the point (x, y, z) (being in the most general case a variable), we have (Art. 153)

$$dM = \rho\,dV,$$

and the equations may be written

$$\bar{x} = \int x\rho\,dV/\int \rho\,dV; \quad \bar{y} = \int y\rho\,dV/\int \rho\,dV;$$
$$\bar{z} = \int z\rho\,dV/\int \rho\,dV. \quad \ldots \quad (2)$$

Homogeneous body.—If the body is homogeneous, ρ is constant, and the formulas reduce to

$$\bar{x} = \int x\,dV/\int dV; \quad \bar{y} = \int y\,dV/\int dV; \quad \bar{z} = \int z\,dV/\int dV. \quad (3)$$

These are also the formulas for the coördinates of the centroid of a volume.

Any area.—Starting with the formulas for the coördinates of the centroid of an area already given (Art. 155), let the number of partial areas be increased while their size decreases, approaching zero as a

limit. The values of $\bar{x}$, $\bar{y}$ and $\bar{z}$ approach limiting values identical in form with those given by equations (3). These equations may therefore be used in finding the centroid of an area if dV represents an element of area. In case of a plane area, only two of the equations are needed, the plane of the area being taken as the plane of two of the coördinate axes.

Any line.—The values of $\bar{x}$, $\bar{y}$ and $\bar{z}$ given by equations (3) may be regarded as the coördinates of the centroid of any line, straight or curved, if dV is taken to represent an element of length. For a plane curve only two of the equations are needed, and for a straight line but one equation is needed. The centroid of a straight line is evidently at its middle point.

160. Methods of Solution.— In applying the general formulas for center of mass, center of volume, center of area and center of length to particular problems, many special methods are found useful. Thus, the integration may be double, single, or triple, depending not only upon the number of dimensions of the body considered, but also upon the way in which the differential element is taken. Again, it is often convenient to employ polar coördinates, or to otherwise change the variable with respect to which the integration is to be performed. These various devices are best explained in connection with the problems in which they are employed. Several such problems will now be solved.

161. Applications.—I. To determine the centroid of a circular arc.

Let AB (Fig. 88) be the given arc, the radius being r. By symmetry it is seen that the centroid lies upon OX, drawn through the center bisecting the angle AOB. Let θ denote the angle between OX and the radius vector drawn from O to any point of the arc. If s represents the length of the arc, measured from some fixed point, ds replaces dV in the general formula for $\bar{x}$; that is,

$$\bar{x} = \int x\,ds / \int ds.$$

Introducing θ as the variable,

$$x = r\cos\theta, \quad ds = r\,d\theta.$$

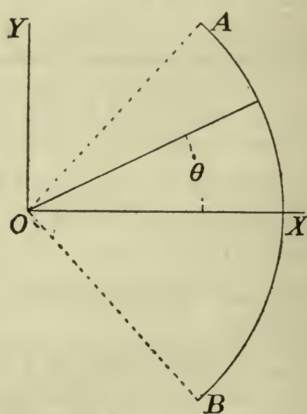


FIG. 88.

If the angle AOB is α, the limits of integration are $\alpha/2$ and $-\alpha/2$. Hence

$$\bar{x} = \frac{\int_{-\alpha/2}^{\alpha/2} r^2 \cos\theta \, d\theta}{\int_{-\alpha/2}^{\alpha/2} r \, d\theta}.$$

But $\int_{-\alpha/2}^{\alpha/2} \cos\theta \, d\theta = 2 \sin \alpha/2$; and $\int_{-\alpha/2}^{\alpha/2} d\theta = \alpha$; hence

$$\bar{x} = \frac{2r \sin(\alpha/2)}{\alpha} = \frac{r \sin(\alpha/2)}{\alpha/2}.$$

II. To find the centroid of the area of a triangle.

Let ABC (Fig. 89) be the triangle. Draw AX perpendicular to BC, and let u denote the length of $B'C'$, drawn parallel to BC at a distance x from A. Let a denote the length of BC, and h the perpendicular distance from A to BC; then the value of u is ax/h. In the formula $\bar{x} = \int x\,dV / \int dV$ (Art. 159) there may be substituted

$$dV = u\,dx = (a/h)\,x\,dx;$$

and therefore

$$\bar{x} = \frac{\int_0^h x^2\,dx}{\int_0^h x\,dx} = \frac{2}{3}h.$$

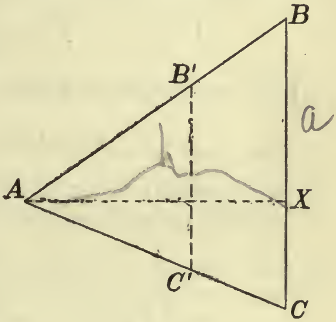


FIG. 89.

Since any vertex of the triangle may be used instead of A, the above result shows that *the centroid of a triangular area is at a distance from either side equal to one-third the altitude measured from that side.*

If a line is drawn from each vertex bisecting the opposite side, the three lines will intersect in a point which may be proved geometrically to coincide with the centroid as above determined. (See Art. 158.)

III. To find the centroid of the area of a circular quadrant.

Let a be the radius of the circle, and let OX and OY (Fig. 90) coincide with the radii bounding the quadrant. Either single or double integration may be employed, as will be shown.

Single integration.—In the formula for $\bar{x}$ the element of area may be any element, all parts of which have the same value of x. Hence we may take

$$dV = y\,dx,$$

in which y is the ordinate of the curve at the point whose abscissa is x; or, expressing y in terms of x by means of the equation of the circle,

$$dV = (a^2 - x^2)^{1/2}\,dx.$$

Hence the formula for $\bar{x}$ becomes

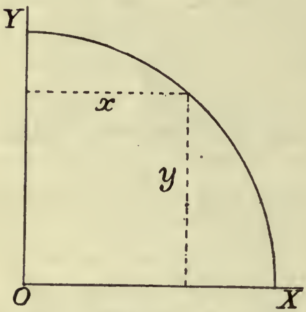


FIG. 90.

$$\bar{x} = \frac{\int_0^a (a^2 - x^2)^{1/2} x\,dx}{\int_0^a (a^2 - x^2)^{1/2}\,dx}.$$

Reducing numerator and denominator,

$$\int_0^a (a^2 - x^2)^{1/2} x\,dx = \Big[-\tfrac{1}{3}(a^2 - x^2)^{3/2}\Big]_0^a = a^3/3;$$

$$\int_0^a (a^2 - x^2)^{1/2}\,dx = \tfrac{1}{2}\Big[x(a^2 - x^2)^{1/2} + a^2 \sin^{-1}(x/a)\Big]_0^a = \pi a^2/4;$$

$$\bar{x} = (a^3/3)/(\pi a^2/4) = 4a/3\pi.$$

From symmetry it is evident that $\bar{y} = \bar{x}$.

Double integration.—Let the element of area be taken as

$$dV = dx\,dy.$$

Then

$$\bar{x} = \iint x\,dx\,dy \,/\! \iint dx\,dy; \quad \bar{y} = \iint y\,dx\,dy \,/\! \iint dx\,dy.$$

(Notice that x and y now denote the coördinates of any element $dx\,dy$, not the coördinates of the curve.) The limits of the integrations with respect to the two variables will depend upon which integration is performed first. If the x-integral is first taken, the limits for x are o and $\sqrt{a^2 - y^2}$; since for *any* value of y these are the extreme values of x for the area in question. The integration with respect to y would then have the limits o and a. Thus,

$$\bar{x} = \frac{\int_0^a \int_0^{\sqrt{a^2-y^2}} x\,dy\,dx}{\int_0^a \int_0^{\sqrt{a^2-y^2}} dy\,dx}.$$

Performing the x-integration,

$$\bar{x} = \frac{\int_0^a (a^2 - y^2)dy}{2\int_0^a \sqrt{a^2 - y^2} \cdot dy}.$$

Taking the y-integral,

$$\bar{x} = (a^3/3)/(\pi a^2/4) = 4a/3\pi.$$

The value of $\bar{y}$ may be found in a similar manner, but is evidently equal to $\bar{x}$.

The above solutions are sufficient to illustrate the general method to be adopted in determining centroids by integration. The best choice of coördinates and of elementary volumes, areas or lengths, is a matter calling for mathematical ingenuity. The matter of fundamental importance to the student is a clear understanding of the general method, rather than the ability to solve special problems by memory.

EXAMPLES.

1. Determine by integration the centroid of the volume of any pyramid or cone.

2. Determine the centroid of the area bounded by the parabola $y^2 = 2px$, the axis of x, and any ordinate.

3. Find the centroid of the area bounded by the parabola, the axis of x, and any two ordinates.

4. Find the centroid of the volume bounded by the surface of a sphere of radius a and two parallel planes distant x_1 and x_2 from the center.

Ans. $\bar{x} = [3(x_1 + x_2)(2a^2 - x_1^2 - x_2^2)]/[4(3a^2 - x_1^2 - x_1x_2 - x_2^2)]$.

5. Find the centroid of the area of any zone of a sphere.

6. Determine the centroid of the area bounded by a circle and any two parallel lines.

7. From a circular area of radius r, a second circular area of radius r' is removed. The distance between the centers being c, determine the centroid of the area remaining.

8. Determine the centroid of the area common to two circles of radii r and r', the center of the latter lying on the circumference of the former.

9. Determine the centroid of the volume of a hemisphere.

Ans. At a distance from the center equal to three-eighths the radius.

10. Determine the center of mass of a hemisphere in which the density varies directly as the distance from the center.

Ans. At a distance from the center equal to two-fifths the radius.

11. Determine the center of mass of a sphere made up of two hemispheres, each homogeneous, but one twice as dense as the other.

12. Determine the position of the centroid of the surface of a right cone.

CHAPTER X.

FORCES IN THREE DIMENSIONS.

§ 1. *Concurrent Forces.*

162. Resultant of Any Number of Concurrent Forces.— The resultant of any number of concurrent forces, whether coplanar or not, may be found by the repeated application of the principle of the parallelogram or triangle of forces. The resultant of any two is a single force equal to their vector sum. This resultant, combined with a third force, gives as the resultant of the three a single force equal to their vector sum; and so on. That is, the resultant of any number of non-coplanar concurrent forces is a single force equal to their vector sum and acting at their common point of application.

163. Parallelopiped of Forces.— Any three non-coplanar concurrent forces and their resultant may be represented in magnitude and direction by three edges and a diagonal of a parallelopiped.

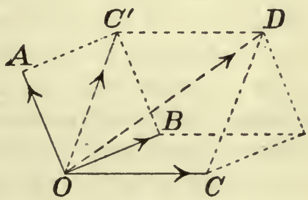


FIG. 91.

Thus, let OA, OB and OC (Fig. 91) represent, in magnitude and direction, three non-coplanar forces. The resultant of OA and OB is represented by OC', the diagonal of the parallelogram $OAC'B$. The resultant of OC' and OC is represented by OD, the diagonal of the parallelogram $OC'DC$. But OD is the diagonal of the parallelopiped of which OA, OB and OC are adjacent edges.

164. Resolution of a Force into Three Components.— A force may be resolved into three components acting in any three non-coplanar lines passing through a point in its line of action. For if the given force be represented in magnitude and direction by a vector, a parallelopiped may always be determined of which this vector is a diagonal, and whose edges are parallel to the three chosen lines.

The resolution of a given force into three components acting in three chosen non-coplanar lines can be made in but one way; for a parallelopiped is completely determined if a diagonal is given in

length and direction, and if the directions of the three adjacent edges are also given.*

If the three components are mutually perpendicular, their magnitudes may be simply computed from the angles they make with the resultant. Thus, if the parallelopiped in Fig. 91 is rectangular, and if α, β, γ are the angles between OD and the three edges OA, OB, OC, we have

$$OA = OD \cos \alpha; \quad OB = OD \cos \beta; \quad OC = OD \cos \gamma.$$

165. Computation of Resultant of Concurrent Forces.— The simplest method of determining the magnitude and direction of the resultant of non-coplanar concurrent forces is usually to replace each force by three rectangular components, and then combine these components.

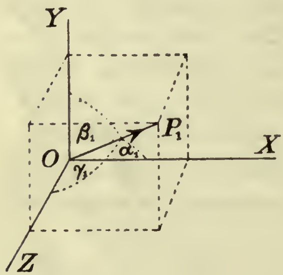


FIG. 92.

Thus, having given any concurrent forces, of magnitudes P_1, P_2, . . . , choose a set of rectangular axes OX, OY and OZ (Fig. 92), and let the angles made by P_1 with these three axes respectively be α_1, β_1, γ_1, with similar notation for the other forces. The axial components of P_1 are $P_1 \cos \alpha_1$, $P_1 \cos \beta_1$, $P_1 \cos \gamma_1$; and similar expressions may be written for the components of every force. The whole system is thus reduced to three sets of collinear forces, which combine into three forces as follows:

a force $P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots$ parallel to OX;
" " $P_1 \cos \beta_1 + P_2 \cos \beta_2 + \ldots$ " " OY;
" " $P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \ldots$ " " OZ.

Let R denote the magnitude of the resultant; a, b, c its angles with OX, OY, OZ; and X, Y, Z its axial components. Then

$$X = R \cos a = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots;$$
$$Y = R \cos b = P_1 \cos \beta_1 + P_2 \cos \beta_2 + \ldots;$$
$$Z = R \cos c = P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \ldots;$$
$$R^2 = X^2 + Y^2 + Z^2;$$
$$\cos a = X/R; \quad \cos b = Y/R; \quad \cos c = Z/R.$$

*If the three components are coplanar with the given force, the problem is indeterminate. If only two of the components are coplanar with the given force, the third component is zero.

166. Equilibrium. — The general condition of equilibrium for any system of concurrent forces is that the resultant is zero.

From the equation

$$R = \sqrt{X^2 + Y^2 + Z^2} = 0$$

may be derived the three equations

$$X = 0, \quad Y = 0, \quad Z = 0.$$

For if these three equations are not satisfied, R cannot equal zero unless either X^2, Y^2 or Z^2 is negative, which would make X, Y or Z imaginary. But neither of these quantities can be imaginary for any real system of forces.

Since the directions of resolution may be assumed at pleasure, it follows that

If a system of concurrent forces is in equilibrium, the sum of their resolved parts in any direction must equal zero.

In accordance with this principle an infinite number of equations may be written. Only three such equations can be independent as the following analysis shows:

If the sum of the resolved parts in one direction is zero, the resultant, if not zero, must act in a plane perpendicular to the direction of resolution. If the sum of the resolved parts is zero for a second direction, the resultant, if not zero, must act in a line perpendicular to the two directions of resolution. If the sum of the resolved parts is zero for a third direction not coplanar with the other two, the resultant must be zero.

It follows that if three equations be formed by resolving in three non-coplanar directions, any equation obtained by resolving in a fourth direction may be derived from these three.

Examples.

1. A particle of 45 lbs. mass is suspended by three strings, each 8 ft. long, attached at points A, B, C in the same horizontal plane, the distances AB, BC, CA being each 4 ft. Determine the tensions in the supporting strings. *Ans.* 16.37 lbs. in each string.

2. A particle of 60 lbs. mass is suspended by three strings, each 12 ft. long, attached at points A, B, C, in the same horizontal plane, the distance AB being 6 ft. and BC and CA each 8 ft. Determine the tensions in the supporting strings.

3. A particle of 75 lbs. mass is suspended by three strings, each 10 ft. long, attached at points A, B, C, in the same horizontal plane,

the distances AB, BC, CA being 4 ft., 5 ft. and 6 ft. respectively. Required the tensions in the supporting strings.

Ans. 33.62 lbs., 9.04 lbs., 35.84 lbs.

4. A body of 5 kilogr. mass is suspended by a string which is knotted at C to two strings CA, CB, these being attached to fixed supports at A and B in a horizontal plane. At C is knotted a fourth string which passes over a smooth peg at D and sustains a mass of 4 kilogr. In the position of equilibrium CD is horizontal and at right angles to AB. The distances AC, BC, AB are 180 c.m., 150 c.m., and 120 c.m. respectively. Determine the position of equilibrium and the tensions in the strings AC, BC.

Ans. Inclination of plane to vertical, 38° 40′. Tension in AC, 1.21 kilogr.; tension in BC, 5.44 kilogr.

§ 2. *Composition and Resolution of Couples.*

167. Representation of Couple by Vector. — It has been shown in Chapter IV that two couples applied to the same rigid body are equivalent if (1) they act in parallel planes, (2) their moments are equal in magnitude, and (3) their rotation-directions coincide. So long as the discussion was limited to coplanar forces, a couple could be completely represented by a quantity whose numerical magnitude specified the value of the moment and whose algebraic sign specified the direction of rotation in the plane. It is now needful to consider couples whose planes are not parallel.

A couple may be completely represented by a *vector* in the following manner: (*a*) The vector is to be taken perpendicular to the plane of the couple; (*b*) its length must represent, to some chosen scale, the magnitude of the moment of the couple; (*c*) its direction must correspond to the rotation-direction of the couple in its plane. These requirements may be further explained as follows:

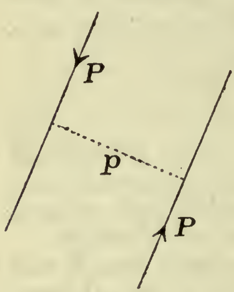


FIG. 93.

Let the plane of the paper be parallel to the plane of the couple; let P denote the magnitude of each force and p the length of the arm; and let the lines of action and directions be as shown in Fig. 93. Then (*a*) the vector which represents the couple must be perpendicular to the plane of the paper; (*b*) its length must be Pp units (the unit length which represents one unit of moment being

chosen arbitrarily); (c) it must be decided which of the two opposite directions perpendicular to the plane of the paper represents the kind of rotation shown in the figure and which the opposite kind. It will here be assumed that the couple shown in the figure is to be represented by a vector pointing upward from the paper. The rule thus assumed may be stated generally as follows:

If a right-hand screw be placed with its axis perpendicular to the plane of the couple, when its direction of rotation coincides with that of the couple its direction of advance will coincide with the direction of the vector representing the couple.

168. Resultant of Any Two Couples.— *The resultant of any two couples is a couple, and the vector representing it is equal to the sum of the vectors representing the two given couples.*

This has already been proved for the case in which the planes of the couples are parallel, the vector sum in this case reducing to the algebraic sum (Art. 93).

If the planes of the two couples are not parallel, let the plane of the paper be perpendicular to their line of intersection, this line being projected at B (Fig. 94), and BM, BN being the traces of the two planes. Let G_1 denote the moment of the couple in the plane BM and G_2 the moment of the couple in the plane BN, their directions of rotation being such that they are represented by the vectors $A'B'$, $B'C'$.

Replace the former couple by an equivalent couple of which the forces are perpendicular to the plane of the paper, P being the magnitude of each force and G_1/P its arm. One of these forces may be taken as acting at B and the other at A, if $AB = G_1/P$. Similarly, the second couple may be replaced by a couple of forces P, one acting at B, opposite to the force P already assumed to act at B, and the other at C, if $BC = G_2/P$. Since the two forces applied at B

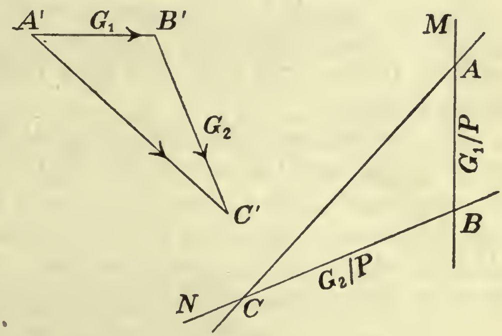


Fig. 94.

balance each other, the two couples are equivalent to a couple of equal and opposite forces P applied at A and C, its moment being $P \times AC$.

It may be shown that this resultant couple is completely represented by the vector $A'C'$. In the triangles $A'B'C'$, ABC, the side $A'B'$ is perpendicular to AB and equal to $P \times AB$; and the side $B'C'$ is perpendicular to BC and equal to $P \times BC$. Therefore $A'C'$ is perpendicular to AC and equal to $P \times AC$, which is the moment of the resultant couple. It is also evident that the direction of the vector $A'C'$ represents correctly the rotation-direction of the resultant couple. Thus the proposition is proved.

169. Resultant of Any Number of Couples; Resolution of a Couple.— By repeated applications of the above principle, any number of non-coplanar couples may be combined; it is seen that the resultant will always be a couple, and that the vector representing it is the sum of the vectors representing the several couples. This may be briefly expressed by saying that *the resultant couple is the vector sum of the given couples.*

By reversing this process, a given couple may be replaced by several component couples.

Since in general a vector may be resolved, in one way only,* into three components whose directions are given, a couple may be resolved (in one way only) into three component couples whose planes are given.

The simplest method of computing the resultant of any number of couples is usually to replace each by its components parallel to three rectangular planes. The whole system is thus reduced to three sets of coplanar couples, each of which reduces to a single couple by algebraic addition of the several moments. If L, M, N are the moments of these partial resultants, the final resultant has the moment

$$G = \sqrt{L^2 + M^2 + N^2}.$$

If l, m, n are the angles between the vector representing the resultant couple and the vectors representing its three components,

$$\cos l = L/G; \quad \cos m = M/G; \quad \cos n = N/G.$$

---

* If the three components are coplanar, the resolution is impossible unless the given vector lies in the plane of the components; in which case the resolution can be made in an infinite number of ways.

§ 3. *Any System of Forces.*

170. Any System Reduced to a Force and a Couple.—*Any system of forces applied to a rigid body is equivalent to a single force and a couple.*

Let P represent the magnitude of any one of the given forces. At any point O, introduce a force equal and parallel to P and an equal and opposite force. One of these, with the given force, forms a couple; so that the given force is equivalent to an equal force P applied at O and a couple. In a similar manner each one of the given forces may be replaced by an equal force applied at O and a couple. The whole system is therefore equivalent to a system of concurrent forces applied at O, equal in magnitude and direction to the given forces, together with a system of couples. The former combine into a single force equal to the vector sum of the given forces; and the latter into a single couple, equal to the vector sum of the several couples.

171. Computation of the Force and Couple.—The single force which results from the above process of combination may be computed as if the forces were concurrent (Art. 165); its magnitude and direction are the same, whatever point is chosen as its point of application. The couple may be computed most readily by a different method of resolution.

Choosing a set of rectangular axes, OX, OY and OZ, let x, y, z be the coördinates of the point of application of any force P, and let P be replaced by its axial components X, Y and Z (Fig. 95). Each of these components may be replaced by an equal force at O and two couples, as follows: Consider the force X applied at A (Fig. 95). Introduce, in the line CB, a force equal to X and an equal and opposite force; and in the line OX also a pair of equal and opposite forces of magnitude X. The force X at A and the equal and opposite force in the line CB form a couple whose plane is parallel to the plane xy and whose moment* is $-Xy$. The remaining force X in the line CB and the equal and opposite force in the line OX

---

*The moments of couples in planes parallel to the yz plane are called plus or minus, according as the vectors representing them point toward the plus or the minus direction of the x-axis. A similar convention is adopted for each of the other coördinate planes.

form a couple lying in the plane zx, of moment Xz. These two couples, with the remaining force in the line OX, are equivalent to the given force X applied at A.

In a similar manner it may be shown that the force Y (Fig. 95) is equivalent to an equal force applied at O; a couple parallel to the plane yz, of moment $-Yz$; and a couple parallel to the plane xy, of moment Yx. Also, Z (Fig. 95) is equivalent to an equal force applied at O; a couple parallel to the plane zx, of moment $-Zx$; and a couple parallel to the plane yz, of moment Zy.*

Combining the three forces at O into a single force, and the six couples into three, it is seen that the given force P, applied at A,

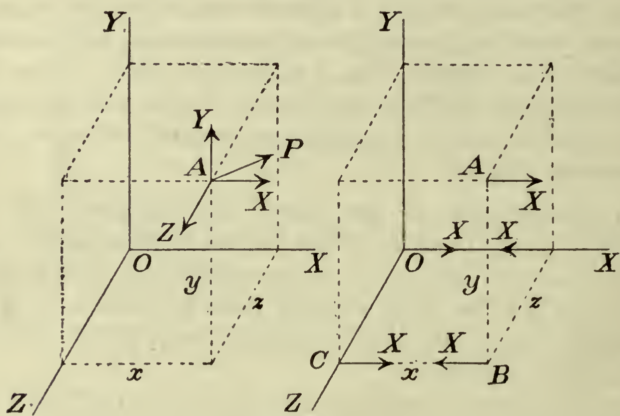


FIG. 95.

is equivalent to an equal and parallel force applied at O, together with three couples as follows:

In the yz-plane, a couple of moment $Zy - Yz$.

In the zx-plane, a couple of moment $Xz - Zx$.

In the xy-plane, a couple of moment $Yx - Xy$.

These three couples may be represented by vectors having the directions OX, OY and OZ, respectively, and of lengths representing the values of the moments.

Let there be given any number of forces, P_1 applied at the point

* These results may be obtained without actually repeating the process employed in case of the force X, by cyclic substitution of the letters xyz and XYZ.

(x_1, y_1, z_1), P_2 applied at the point (x_2, y_2, z_2), etc.; and let each be replaced by a force and three couples in the manner described. Let $L_1 = Z_1y_1 - Y_1z_1$; $M_1 = X_1z_1 - Z_1x_1$; $N_1 = Y_1x_1 - X_1y_1$; with similar notation for each force. The whole system is equivalent to a force R, applied at O, equal to the vector sum of the given forces; and a couple which is the resultant of three couples as follows:

In the yz-plane, a couple of moment $L_1 + L_2 + \ldots = L$.
In the zx-plane, a couple of moment $M_1 + M_2 + \ldots = M$.
In the xy-plane, a couple of moment $N_1 + N_2 + \ldots = N$.

The moment of the resultant couple is

$$G = \sqrt{L^2 + M^2 + N^2};$$

and the direction of its plane may be found as in Art. 169.

172. **Resultant of any System of Forces.**— Having reduced a system of forces to a force and couple as above, if it is attempted to reduce it to a still simpler system, it appears that in certain cases it may be reduced to a couple and in other cases to a single force; but that in general the simplest system to which it can be reduced consists of two non-coplanar forces.

(*a*) If the vector sum of the given forces is 0, the resultant is evidently a couple, determined as above.

(*b*) If the plane of the couple found by the above process is parallel to the single force R, this force and couple are equivalent to a single force (Art. 94) which is the resultant of the given system. This resultant force is equal to the vector sum of the given forces, and its line of action can be determined as in Art. 94.

(*c*) If the plane of the couple is not parallel to the force R, let it be replaced by an equivalent couple of which one force acts in a line intersecting the line of action of R. This force may be combined with R, thus reducing the system to two non-coplanar forces. This reduction to two forces may be made in an infinite number of ways, but in general no further reduction is possible.

§ 4. *Equilibrium.*

173. **Equations of Equilibrium.**—Let a system of forces applied to the same rigid body be reduced to a force and couple as in Art. 171. Let R denote the magnitude of the force, a, b, c the

angles it makes with the coördinate axes, and X, Y, Z its axial components; and let G denote the moment of the couple, l, m, n the angles made with the axes by the vector representing the couple, and L, M, N the moments of the three component couples parallel to the coördinate planes.

In order that the system may be in equilibrium, both R and G must reduce to zero; otherwise the force and couple may be reduced (as in Art. 172) either to a single force, to a couple, or to two non-coplanar forces.

The condition $R = 0$ requires that

$$X = 0; \quad Y = 0; \quad Z = 0; \quad . \quad . \quad . \quad (1)$$

and the condition $G = 0$ requires that

$$L = 0; \quad M = 0; \quad N = 0. \quad . \quad . \quad . \quad (2)$$

For unless X, Y and Z are severally equal to zero, either X^2, Y^2 or Z^2 must be negative, and X, Y or Z therefore imaginary. But from the manner of computing these quantities it is evident that they must be real. Similar reasoning applies to L, M and N.

Equations (1) and (2) are therefore six independent equations of equilibrium. Since the position of the origin may be chosen at pleasure, equations (2) may be written in an infinite number of ways. Also, since the directions of the axes may be chosen arbitrarily, equations (1) may be written in an infinite number of ways.

In order that the conditions of equilibrium may be stated in a concise form, it is convenient to generalize the definition of moment of a force in the following manner.

174. Moment of a Force About an Axis.— The moment of a force with respect to an axis has been defined, for the case in which the axis is perpendicular to a plane containing the line of action of the force (Art. 70), as the product of the magnitude of the force into the perpendicular distance of its line of action from the axis. Thus, if the line of action of the force P lies in the plane of the paper (Fig. 96) and the axis is perpendicular to that plane at O, the moment of the force is equal to Pp.

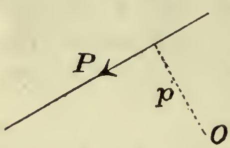


FIG. 96.

If the axis is not perpendicular to the force (that is, to any plane containing the line of action of the force), the moment is to be com-

puted by replacing the force by two components, one parallel and the other perpendicular to the axis, and taking the moment of the latter.

175. Conditions of Equilibrium.— Referring to the process by which a system is reduced to a single force and a couple (Art. 171), it is seen that $Z_1y_1 - Y_1z_1$ is equal to the moment of the force P_1 with respect to the axis OX; $X_1z_1 - Z_1x_1$ to its moment with respect to OY; and $Y_1x_1 - X_1y_1$ to its moment with respect to OZ. The quantity L is therefore equal to the sum of the moments of the given forces about the axis OX; M to the sum of their moments about OY; and N to the sum of their moments about OZ.

The meaning of the two sets of equations of equilibrium (Art. 173) may therefore be stated in words as follows:

(1) The sum of the resolved parts of the given forces parallel to each of the axes is zero.

(2) The sum of the moments of the given forces about each of the given axes is zero.

Since any line whatever may be taken as one of the axes, it follows that, for equilibrium,

(*a*) The sum of the resolved parts of the forces in any direction is zero.

(*b*) The sum of their moments about any axis is zero.

These principles lead to an infinite number of equations; but only six can be independent.

Examples.

1. A three-legged stool rests upon a smooth horizontal floor. Determine the pressures at the three points of support.

2. The three points of support of a three-legged stool being at any distances apart, what must be the position of its center of gravity in order that the pressures at the points of support may be equal?

3. A four-legged table rests upon a smooth horizontal floor. How much can be determined about the supporting forces?

4. A windlass consists of a circular cylinder supported at two points in bearings which permit free revolution about its axis, a crank attached to one end of the cylinder, and a cord wound upon the cylinder and having one end free. To the free end of the cord is attached a heavy body which is to be lifted by the application of a force to the crank-handle. The axis of revolution of the cylinder is horizontal. Assuming that the force applied to the crank is perpendicular both to the axis of revolution and to the crank-arm, and that

the bearings are frictionless, it is required to determine all forces acting on the windlass when the weight of the lifted body is known.

Let $W =$ weight of body lifted; $P =$ force applied to crank-handle; $r =$ radius of cylinder; $a =$ length of crank-arm (*i. e.*, radius of circle described by point of application of P); $l =$ distance between centers of bearings; $b =$ distance from center of circle described by point of application of P to nearest bearing; l_1 and $l_2 =$ distances from centers of bearings to suspended cord. The bearings being assumed smooth, the supporting forces exerted by them may have any directions perpendicular to the axis of the cylinder; let them be replaced by horizontal and vertical components, and let H_1, V_1 denote the components of the force at the bearing near the crank and H_2, V_2 the components of the force at the other bearing. The direction of the force P, if always perpendicular to the

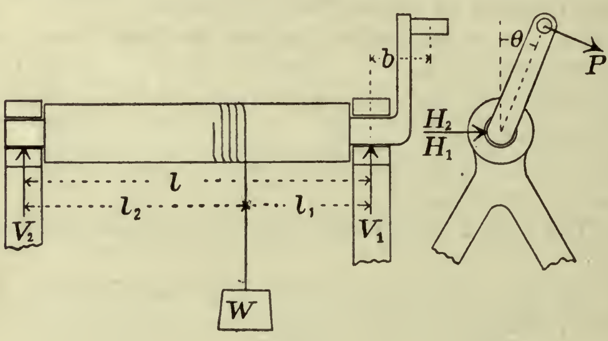


FIG. 99.

crank-arm, will vary as the cylinder revolves. Let $\theta =$ angle between crank-arm and the vertical.

All forces acting upon the windlass act in planes perpendicular to the axis of revolution; hence, if forces are resolved parallel to this axis, no equation results. Five independent statical equations may, however, be written as follows:

Resolving horizontally,

$$H_1 + H_2 + P \cos \theta = 0. \quad . \quad . \quad . \quad (1)$$

Resolving vertically,

$$V_1 + V_2 - P \sin \theta - W = 0. \quad . \quad . \quad (2)$$

Taking moments about the axis of revolution,

$$Pa - Wr = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Taking moments about a vertical line through center of circle described by point of application of P,

$$H_1 b + H_2(b + l) = 0. \quad . \quad . \quad . \quad (4)$$

Taking moments about a horizontal diameter of same circle,

$$V_1 b + V_2(b + l) - W(b + l_1) = 0. \quad . \quad . \quad (5)$$

These five equations contain five unknown quantities, H_1, V_1, H_2, V_2, P. Their solution is simple. Thus, (3) determines P at once; (1) and (4) then determine H_1 and H_2; (2) and (5) determine V_1 and V_2.

5. In Ex. 4, take numerical data as follows: $l_1 = l/3$; $b = l/10$; $a = 6r$. Solve for $\theta = 0$, $90°$, $180°$, $270°$, $45°$.

Ans. $P = W/6$ for any value of θ. For $\theta = 0$, $H_1 = -(11/60)W$, $H_2 = (1/60)W$, $V_1 = (2/3)W$, $V_2 = (1/3)W$. For $\theta = 90°$, $H_1 = H_2 = 0$, $V_1 = (51/60)W$, $V_2 = (19/60)W$.

6. In Ex. 4, substitute for the force P a weight Q suspended from the crank-handle. If Q is known, determine the position of equilibrium and all unknown forces.

7. Assume dimensions as in Ex. 5, and let the weight Q, applied as in Ex. 6, be equal to W. Determine position of equilibrium and all forces.

Ans. $\theta = 9° \, 36'$ or $170° \, 24'$; $H_1 = H_2 = 0$; $V_1 = (17/20)W$; $V_2 = (19/60)W$.

8. A circular cylinder is placed with axis horizontal, one end resting in a smooth cylindrical bearing and the other on an inclined timber. The longitudinal axis of the timber lies in a plane perpendicular to the axis of the cylinder and is inclined to the horizontal at a known angle. If its surface is so rough as to prevent sliding, can the cylinder be in equilibrium?

9. In Ex. 8, can equilibrium be produced by hanging a weight from a string wound on the surface of the cylinder, assuming sliding to be impossible? If equilibrium is thus possible, specify the required weight, and all forces acting on the cylinder.

10. Two timbers are placed with longitudinal axes parallel and with upper surfaces in the same horizontal plane. A circular cylinder rests upon the timbers. Can the supporting timbers be tilted without destroying the equilibrium of the cylinder, assuming the surfaces to be sufficiently rough to prevent sliding?

11. The points of support of a three-legged stool form an equilateral triangle ABC of side a. The center of gravity is vertically above the centroid of the triangle. The stool is pulled horizontally by a string lying in the vertical plane containing AB and at a distance h above the floor. Show that the stool will tip without sliding if the coefficient of friction is greater than $a/3h$.

12. A uniform straight bar is placed with one end on a rough horizontal floor and the other in the right angle between two smooth vertical walls. Show that it will be in equilibrium if its inclination

to the vertical is less than $\tan^{-1}(2\mu)$, μ being the coefficient of friction.

13. In Ex. 12, show that the resultant force acting upon the bar at either end acts in the vertical plane containing the bar. Determine these forces.

14. A bar is placed with one end upon a rough horizontal floor and the other against a rough vertical wall. Determine the positions of limiting equilibrium.

CHAPTER XI.

GRAVITATION.

§ 1. *Attraction Between Two Particles.*

176. Law of Gravitation.— Every portion of matter exerts an attractive force upon every other portion. The magnitude and direction of this attractive force for any two bodies may be determined in accordance with Newton's law of universal gravitation, which may be stated as follows :

Every particle of matter attracts every other particle with a force which acts along the line joining the two particles, and whose magnitude is proportional directly to the product of their masses and inversely to the square of the distance between them.

The proportionality expressed in this law may be stated algebraically as follows : Let P denote the magnitude of the attractive force between two particles whose masses are m_1, m_2, and whose distance apart is r ; and P' the force between two particles whose masses are m_1', m_2', and whose distance apart is r'. Then

$$P/P' = (m_1 m_2/r^2)/(m_1' m_2'/r'^2).$$

This equation may be written in the form

$$Pr^2/m_1 m_2 = P'r'^2/m_1' m_2' = \gamma.$$

Since a similar equation may be written for any pair of particles whatever, the law of gravitation may be expressed by the equation

$$Pr^2/m_1 m_2 = \gamma,$$

or

$$P = \gamma m_1 m_2/r^2,$$

in which γ is a constant.

The value of γ is to be determined by experiment ; but having been determined for one case it is known for all cases.

This formula applies strictly only to particles, but it gives, to a close approximation, the attraction between two bodies of finite size whose linear dimensions are small compared with the distance between them. We shall first consider the attraction between two

particles, and shall then give a brief discussion of methods of computing accurately the attraction between bodies of finite size.

Attraction is a *mutual* action between two particles or bodies; *i. e.*, each exerts an attractive force upon the other, the two forces being equal in magnitude and opposite in direction. This is implied in the above statement of the law of gravitation. It is also in accordance with the law of "action and reaction," Newton's third law of motion (Art. 35).

177. The Constant of Gravitation.—The quantity γ in the above formula is called the *constant of gravitation*. Its numerical value depends upon the units in which force, mass and distance are expressed. For any given system of units, γ is numerically equal to the attractive force between two particles of unit mass at unit distance apart; for if $m_1 = 1$, $m_2 = 1$ and $r = 1$, the formula gives $P = \gamma$.

If mass, length and force be expressed in any units employed in ordinary practical problems, the numerical value of γ is very small. Thus if mass is expressed in pounds and distance in feet, γ is equal to the attractive force between two particles of one pound mass each, placed one foot apart. This force is too small to be detected by ordinary methods of measuring forces; and if expressed in terms of any unit in common use its numerical value is very small. (See Arts. 184, 186.)

178. Gravitation Unit of Mass.—Instead of choosing the units of mass, length and force independently, they may be so chosen as to give γ any desired value. Moreover, two of these units may still be chosen arbitrarily, the third being then so taken as to satisfy the equation $P = \gamma m_1 m_2 / r^2$ with the desired value of γ.

In order to simplify the equation, let $\gamma = 1$. Then the units must be so chosen that two particles, each of unit mass, placed at the unit distance apart, attract each other with unit force. The unit mass which satisfies this condition, the other units having been chosen at pleasure, is called the *gravitation unit*.

To determine the value of this unit of mass in terms of the pound, gram, or other known unit, requires the same process of experiment and reasoning which is involved in the determination of the constant γ when all the units have been chosen. This subject will be resumed in Art. 185.

EXAMPLES.

1. If two particles 20 ft. apart attract each other with a force of 12 lbs., with what force will they attract each other if placed 50 ft. apart?

2. If two particles of 1,000 grams and 12,000 grams mass respectively attract each other with a force equal to the weight of P grams when 30 c.m. apart, with what force will two particles of 1,800 grams and 3,000 grams attract each other when 100 c.m. apart?
Ans. 0.0405P grams weight.

3. In Ex. 2, what is the value of the constant γ in terms of P?
Ans. $\gamma = 3P/40{,}000$.

4. With the data of Ex. (2), what is the gravitation unit of mass?
Ans. A mass equal to $200/\sqrt{(3P)}$ grams.

[The answers given to examples 3 and 4 imply that the centimeter is the unit length and the weight of a gram the unit force.]

179. Dimensions of Units.— The formula expressing the law of gravitation for two particles may be written $Pr^2/m_1m_2 = \gamma$. The constant γ is therefore of dimensions (Art. 14)

$$\mathbf{FL^2/M^2}$$

if the units of force, length and mass are all taken as fundamental,— *i. e.*, are all chosen independently (Art. 13).

The law of gravitation, however, itself furnishes a means of making one of these units depend upon the other two. For the equation expressing the law may be written

$$P = m_1m_2/r^2,$$

if the units be so chosen as to satisfy the condition stated in Art. 178. The relation between the units will then be expressed by the dimensional equation,

$$\mathbf{F = M^2/L^2}.$$

Any two of these units being chosen arbitrarily, the dimensions of the third are given by this equation.

Thus, if, as in Art. 178, the units of force and length are made fundamental, the dimensions of the unit mass are expressed by the equation

$$\mathbf{M = LF^{1/2}}.$$

The relation among units which is expressed by a dimensional equation is thus, to a certain extent at least, arbitrary. This will be further illustrated when the kinetic or "absolute" system of units is explained. (See Art. 219.)

§ 2. *Attractions of Spheres and of Spherical Shells.*

180. Law of Gravitation Applied to Continuous Bodies.—The law of attraction is stated above as applying to *particles*, and these are treated as bodies of finite mass but without finite size. If a body consisted of a finite number of such particles, its resultant attraction for any other body or particle would be computed by finding the resultant of the attractive forces due to all its individual particles.

If, instead of being made up of discrete particles of finite mass, a body occupies space continuously (Art. 5), so that any portion whose mass is finite is of finite volume, the resultant attraction of one body upon another may be computed in a similar manner, but the process involves integration.

Let m_1 and m_2 denote the total masses of two continuous bodies, and Δm_1, Δm_2 any small elements of these masses. If the dimensions of these elementary portions are small in comparison with their distance apart, the attractive force exerted by each upon the other has approximately the value

$$\gamma \Delta m_1 \Delta m_2 / r^2,$$

r being the distance between the elements. If the attraction of every element of m_1 upon every element of m_2 be computed approximately in the same manner, the resultant of these forces will be an approximate value of the resultant attraction of m_1 upon m_2. The approximation is closer the smaller the elements into which the bodies are subdivided; the exact value is the limit approached by the approximate value as the elements approach zero in size. The magnitude of every component force thus approaches zero and the number of components approaches infinity, but their resultant has in general a finite value. The computation of this value involves a process of integration.

In applying this method only a few simple cases will be here considered.

181. Attraction of a Homogeneous Spherical Shell Upon an Interior Particle.—*Proposition.*—The resultant attraction exerted by a homogeneous spherical shell of uniform thickness upon a particle within its inner surface is zero.

Let O (Fig. 98) be the position of the particle, and through O

draw any straight line intersecting the outer surface of the shell in two points A and B. On the outer surface of the shell take an elementary area containing the point A; from every point of the perimeter of this element draw a line through O, and prolong it to intersect the spherical surface at a point near B. We thus get a cone cutting from the spherical surface at B a second elementary area. Call the areas of the elements at A and B, a and b respectively, and let h be the thickness of the shell. The volumes cut from the shell by the two branches of the conical surface are approximately

$$ha \quad \text{and} \quad hb$$

respectively, and their ratio has approximately the value

$$ha/hb = a/b,$$

the approximation being closer as the values of h, a and b are taken smaller. It may be shown that as a and b approach 0,

$$a/b \text{ approaches } \overline{OA}^2/\overline{OB}^2.$$

The tangent planes to the sphere at A and B are equally inclined to the line AB; hence in the limit, as a and b approach 0, they are proportional to their orthographic projections upon a plane perpendicular to AB; that is, to the right sections at A and B of the two branches of the conical volume whose vertex is at O. But these right sections are directly proportional to $\overline{OA}^2$ and $\overline{OB}^2$; hence the areas a and b, the elementary volumes ha and hb, and the masses of these elementary volumes, are proportional to $\overline{OA}^2$ and $\overline{OB}^2$.

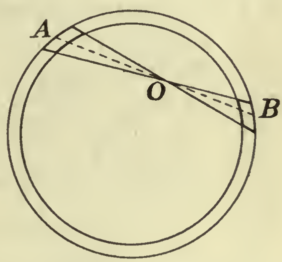


FIG. 98.

If m is the mass of the particle at O, and if m' and m'' are the masses of the elements of the shell at A and B respectively, the particle is attracted toward A with a force

$$\gamma m m'/\overline{OA}^2,$$

and toward B with a force

$$\gamma m m''/\overline{OB}^2.$$

But, as just shown,

$$m'/m'' = \overline{OA}^2/\overline{OB}^2,$$

or

$$m'/\overline{OA}^2 = m''/\overline{OB}^2;$$

hence the attractive forces of m' and m'' upon m are equal and opposite, and their resultant is zero.

The whole volume of the shell may be divided into elements which, taken in pairs, are related in the same way as the two elements considered. Hence the resultant attraction of the shell upon the particle at O is zero.

In the above reasoning, it was assumed that the thickness of the shell becomes small, approaching zero as a limit. If the thickness has any finite value the result still holds. For the shell may be regarded as made up of a great number of very thin shells for each of which the conclusion is true at least approximately; and if the number of shells is increased without limit, the thickness of each approaching zero, the conclusion holds strictly for each elementary shell and therefore for the given shell.

The proposition is also true if the density of the shell varies in such a way that its value is the same at all points equally distant from the center of the sphere.

182. Attraction of a Homogeneous Shell Upon an Exterior Particle.— It is evident from symmetry that the resultant attraction of a homogeneous spherical shell of uniform thickness upon an exterior particle is directed along the line joining the particle with the center of the sphere.

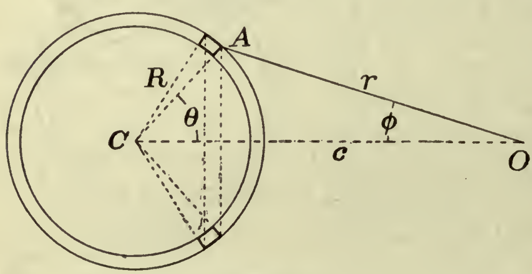


FIG. 99.

Let A (Fig. 99) be any point of the surface of the shell, C the center of the sphere, and O the position of the exterior particle, its mass being m. Take an element of the spherical surface containing A, and let a denote the area of this element, h the thickness and ρ the density of the shell, R the radius of the sphere. Let $OA = r$, $OC = c$, angle $ACO = \theta$, angle $AOC = \phi$.

The conical surface whose base is a and apex C cuts from the shell an element of volume whose mass is ρha. The attraction of

this mass upon the particle at O is $\gamma mpha/r^2$, the resolved part of which in the direction OC is

$$(\gamma mpha/r^2) \cos \phi.$$

Let the element of area a be taken as part of a zone of infinitesimal width, included between two circles whose planes are perpendicular to OC. For every point of such a zone, r and ϕ have the same values; hence the total attraction in the direction OC due to the portion of the shell corresponding to such a strip is

$$\gamma mph \cos \phi \, dA/r^2,$$

if dA represents the area of the zone. The attraction due to the whole shell is therefore

$$P = \gamma mph \int \frac{\cos \phi \, dA}{r^2},$$

the integration being so taken as to cover the whole surface of the sphere. We have now to express ϕ, r and dA in terms of a single variable and integrate. Let r be the variable chosen.

We have

$$dA = Rd\theta \cdot 2\pi R \sin \theta = 2\pi R^2 \sin \theta \, d\theta.$$

But $\quad r^2 = c^2 + R^2 - 2cR \cos \theta; \quad r\,dr = cR \sin \theta \, d\theta;$

hence $\quad dA = (2\pi R/c) r\,dr.$

Again, $\quad R^2 = c^2 + r^2 - 2cr \cos \phi,$

or $\quad \cos \phi = (c^2 + r^2 - R^2)/2cr.$

Substituting the values of $\cos \phi$ and dA in the value of P, and expressing the proper limits,

$$P = \frac{\pi \gamma mphR}{c^2} \int_{c-R}^{c+R} \frac{c^2 - R^2 + r^2}{r^2} dr. \quad . \quad . \quad (1)$$

The integral expression can be separated into two parts, thus:

$$\int \frac{c^2 - R^2 + r^2}{r^2} dr = (c^2 - R^2) \int \frac{dr}{r^2} + \int dr$$

$$= -(c^2 - R^2)\frac{1}{r} + r.$$

Taking the value between the proper limits,

$$\int_{c-R}^{c+R} \frac{c^2 - R^2 + r^2}{r^2} dr =$$

$$-[(c - R) - (c + R)] + [(c + R) - (c - R)] = 4R.$$

Hence $\qquad P = \pi\gamma m\rho hR \cdot 4R/c^2 = \gamma \cdot 4\pi R^2 h\rho \cdot m/c^2.$

But $4\pi R^2 h\rho = M =$ total mass of shell. Hence

$$P = \gamma Mm/c^2; \qquad . \qquad . \qquad . \qquad . \quad (2)$$

which shows that

The resultant attraction between the shell and an exterior particle has the same value as if the entire mass of the shell were concentrated at its center.

The result obviously holds for a shell of any thickness whose density has the same value at all points equally distant from the center. For such a shell may be subdivided into elementary shells, each of which may be taken as thin and as nearly of uniform density as desired.

Interior particle.— The reasoning by which (1) is deduced applies also to the case of an interior particle, except that the limits of the integration must be different. Thus, for the attraction on an interior particle, we have

$$P = \frac{\pi\gamma m\rho hR}{c^2} \int_{R-c}^{R+c} \frac{c^2 - R^2 + r^2}{r^2} dr = 0,$$

which agrees with Art. 181.

183. Attraction of a Sphere Upon a Particle.— The foregoing results may be applied to a sphere whose density has the same value at all points equally distant from the center.

Exterior particle.— The resultant attraction of such a sphere upon an exterior particle has the same value as if the entire mass of the sphere were concentrated at its center.

Interior particle.— The resultant attraction of such a sphere upon an interior particle is the same as if the portion of the mass nearer the center than the particle were concentrated at the center; the remaining shell being disregarded because its resultant attraction is zero.

EXAMPLES.

1. Assuming the earth to be a sphere whose density is a function of the distance from the center, compare the weight of a body at the surface and at a point h feet above the surface.

The "weight" of a body is the force with which the earth attracts it.* If R is the radius of the earth, and if P and P' are the values of the earth's attraction upon the body at the surface and at h feet above the surface respectively, we have

$$P/P' = (R + h)^2/R^2,$$

or

$$P' = PR^2/(R + h)^2.$$

Hence the weight of a body at height h above the surface is the fraction $R^2/(R + h)^2$ of its weight at the surface.

If h is a small fraction of R, we have, approximately,

$$R^2/(R + h)^2 = (1 + h/R)^{-2} = 1 - 2h/R.$$

2. At what height above the surface will the weight of a body be 1 per cent less than at the surface? [The mean radius of the earth is very nearly 6.3709×10^8 c.m. or 20,902,000 ft.]

3. If the pound-force is defined as the weight of a pound-mass at the earth's surface, how much does a change of elevation of 10,000 ft. affect the value of this unit force?

4. Let G denote the weight of a given mass at the earth's surface, and G' its weight at a depth h below the surface. If the earth were a sphere of uniform density and of radius R, show that

$$G'/G = (R - h)/R = 1 - h/R.$$

5. Assuming that the earth is a sphere and that the density is a function of the distance from the center, let ρ denote the mean density of the whole earth and ρ_0 the mean density of the outer shell of thickness h. Determine the relation between the weight of a body at the surface and its weight at a depth h below the surface.

Let M be the mass of the whole earth, M' that of the inner sphere of radius $R - h$, m the mass of the given body, G its weight at the surface and G' its weight when at depth h below the surface. Then G is equal to the attraction between two particles of masses M and m whose distance apart is R; and G' is equal to the attraction between two particles of masses M' and m whose distance apart is $R - h$. That is, $G = \gamma Mm/R^2$, $G' = \gamma M'm/(R - h)^2$. Hence

$$\frac{G'}{G} = \frac{M'}{M}\left(\frac{R}{R-h}\right)^2. \quad . \quad . \quad . \quad . \quad (1)$$

But $\quad M = \frac{4}{3}\pi R^3\rho; \quad M - M' = \frac{4}{3}\pi\rho_0[R^3 - (R - h)^3].$

* The effect of the earth's rotation upon the apparent weight is here disregarded. See Art. 311.

$$\therefore \quad \frac{M'}{M} = 1 - \frac{\rho_0}{\rho}\left[1 - \left(\frac{R-h}{R}\right)^3\right] = \left(1 - \frac{\rho_0}{\rho}\right) + \frac{\rho_0}{\rho}\left(\frac{R-h}{R}\right)^3. \quad (2)$$

Substituting this value in equation (1),

$$\frac{G'}{G} = \left(1 - \frac{\rho_0}{\rho}\right)\left(\frac{R}{R-h}\right)^2 + \frac{\rho_0}{\rho}\left(\frac{R-h}{R}\right). \quad . \quad . \quad . \quad (3)$$

Approximate simple formula.—If h is a small fraction of R, equation (3) may be reduced to the following as a first approximation:

$$\frac{G'}{G} = 1 + \left(2 - 3\frac{\rho_0}{\rho}\right)\frac{h}{R}. \quad . \quad . \quad . \quad . \quad (4)$$

6. Show that, if the mean density of the outer layer of the earth is less than two-thirds the mean density of the whole earth, the weight of a body increases as it is carried below the surface.

This follows from equation (4) above. It will be seen that, as a body is carried below the surface of the earth, its change of weight is the resultant of two opposite effects. The total mass attracting it is less because the outer shell has on the whole no effect (Art. 181); while the attraction of the inner sphere is greater because the body is nearer its center (Art. 183). The latter effect is greater than the former unless the mean density of the outer shell is at least two-thirds that of the whole earth.

According to the best determinations the mean density of the earth is very nearly 5.527 times that of water, while the average density of rocks near the surface may be taken at about half this value. If $\rho_0/\rho = 1/2$, equation (4) becomes

$$G'/G = 1 + h/2R. \quad . \quad . \quad . \quad (5)$$

That the weight of a body increases when it is carried below the surface of the earth has been shown by experiment.

7. What would be the change in the weight of a body if carried 2,000 ft. below the earth's surface, assuming the density of the outer portion of the earth to be half the mean density?

184. Value of the Constant of Gravitation.—If the attraction between two particles of known mass at a known distance apart can be determined by experiment, the value of γ, the constant of gravitation, can be determined. From the foregoing results (Arts. 182, 183), spheres of any size may be used instead of particles. The attraction between bodies of ordinary size is so small that it can be measured only by the most delicate apparatus.

The relation between the constant of gravitation and the mass of the earth may be shown as follows, assuming the earth to be a sphere whose density is a function of the distance from the center. We shall

take as units those of the British gravitation system. The unit mass is the pound, the unit force the weight of a pound mass at the earth's surface (the pound-force), the unit length the foot.

Let R denote the radius of the earth in feet, M its mass, ρ its mean density. Consider the attraction of the earth upon a body of mass m at the surface.

By the general formula of Art. 176 the value of this attraction is $\gamma Mm/R^2$. But expressed in pounds-force it is also equal to m. Hence $m = \gamma Mm/R^2$, or

$$\gamma M = R^2.$$

Since the value of R is known, this equation serves to determine either of the quantities γ and M when the other is known.

Instead of M, we may introduce the earth's mean density ρ, since $M = 4\pi R^3\rho/3$. The equation then becomes

$$\gamma\rho = 3/4\pi R.$$

The best determinations give as the earth's mean density about 345 pounds per cubic foot. Taking $R = 20,900,000$ ft., the value of the constant of gravitation* is

$$\gamma = 3/4\pi R\rho = 3/(4\pi \times 20,900,000 \times 345) = 3.31 \times 10^{-11}.$$

185. Value of Gravitation Unit of Mass.— Let m pounds be the mass of each of two particles which, placed one foot apart, attract each other with one pound-force. The value of m may be found from the formula $P = \gamma m_1 m_2/r^2$, by putting $P = 1$, $r = 1$, $m_1 = m_2 = m$, and $\gamma = 3.31 \times 10^{-11}$ as found above. The result is

$$m = 1/\sqrt{\gamma} = 173,800 \text{ pounds}.$$

If a mass equal to 173,800 pounds be taken as the unit mass, the constant γ becomes unity, and the formula for the attraction between two particles whose masses are m_1 and m_2 and whose distance apart is r is

$$P = m_1 m_2/r^2.$$

Thus, the "gravitation unit of mass" (Art. 178) is a mass equal to about 173,800 pounds, if distance is expressed in feet and force in pounds-force.

---

*The above is not given as the actual method of determining γ, but merely as showing about what its value is. The above value of the earth's mean density is based upon the experimental determination of γ by the measurement of the attraction between bodies of known mass.

186. Values of the Constant of Gravitation and of the Gravitation Unit of Mass in the C. G. S. System.* — If the unit mass is the gram, the unit length the centimeter and the unit force the dyne (Art. 217), the value of γ may be found as follows :

The attraction of the earth for a body of m grams mass at the surface is approximately $981m$ dynes. By the law of gravitation it is also equal to $\gamma Mm/R^2$. Hence $981m = \gamma Mm/R^2$, or

$$\gamma M = 981 R^2.$$

Here M is the mass of the earth in grams, and R is its radius in centimeters. Introducing the earth's mean density ρ instead of M, the equation is

$$\gamma\rho = 3 \times 981/4\pi R.$$

Taking $\rho = 5.527$ grams per cubic centimeter and $R = 6.371 \times 10^8$ centimeters,

$$\gamma = (3 \times 981)/(4\pi \times 5.527 \times 6.371 \times 10^8) = 6.65 \times 10^{-8}.$$

Let m grams be the mass of each of two particles which, when 1 centimeter apart, attract each other with a force of 1 dyne. Then

$$1 = \gamma m^2,$$

or $$m = 1/\sqrt{\gamma} = 1/\sqrt{6.65 \times 10^{-8}} = 3,880 \text{ grams}.$$

If a mass of 3,880 grams is taken as the unit mass, and if the centimeter is the unit length, the attraction in dynes between two particles whose masses are m_1 and m_2 and whose distance apart is r is given by the formula

$$P = m_1 m_2 / r^2.$$

The above relation between the constant of gravitation and the earth's mean density is only approximate, since the earth does not attract bodies at the surface exactly as if its mass were concentrated at its center. The value of the earth's mean density above given is that of Professor C. V. Boys.† It is based upon the value $\gamma = 6.6576 \times 10^{-8}$.

---

* The "absolute" or kinetic system of units is explained in Chapter XII. The present Article presupposes a knowledge of this system.

† See "Nature," Vol. L, p. 419. For values found by other experimenters see "Nature," Vol. LV, p. 296.

PART II.

MOTION OF A PARTICLE.

CHAPTER XII.

MOTION IN A STRAIGHT LINE: FUNDAMENTAL PRINCIPLES.

§ 1. *Position, Displacement and Velocity.*

187. Position of a Particle in a Given Line.— If a particle moves in a given straight line, its position at any instant is known if its distance from a given fixed point in the line is known. Let O (Fig. 100) be the fixed point, and P the position of the particle at any instant. Let the distance OP be represented by x; then the position of the particle is specified by the value of x. The quantity x is called the *abscissa* of the particle.

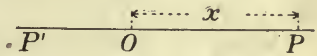


FIG. 100.

The value of x involves direction as well as distance. The two directions OP, OP' along the line of motion are distinguished as plus and minus.

188. Displacement.— The change of position of a particle during any interval of time is called its *displacement.* If A and B (Fig. 101) are its positions at the beginning and end of the interval, AB is its displacement. The displacement, like the abscissa, is called plus or minus according to its direction along the line of motion.

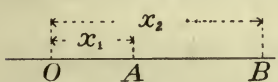


FIG. 101.

If x_1 and x_2 are the values of x at the beginning and end of the interval respectively, $x_2 - x_1$ gives the displacement in magnitude and sign.

189. Uniform Motion.— The motion of a particle is uniform if it receives equal displacements in any equal intervals of time, however small these intervals may be taken.

190. Velocity.— The *velocity* of a particle is its rate of moving (or rate of displacement).

If the particle moves uniformly throughout a certain interval, its velocity is proportional directly to its total displacement during the interval and inversely to the duration of the interval.

The numerical value of the velocity depends upon the unit in terms of which it is expressed.

191. Unit Velocity.—Any "rate of moving" might be chosen as the unit velocity, but it is convenient to choose as the unit the velocity possessed by a uniformly moving particle which passes over the unit distance in the unit time.

The unit thus defined is a derived unit (Art. 13) whose value depends upon the units of distance and time.

With the foot and second as the fundamental units, the unit velocity is the *foot-per-second.* This is the unit ordinarily employed in engineering applications in those countries in which the British system is in common use.

When the French metric system is used, the meter takes the place of the foot as the engineers' unit of length, and the corresponding unit of velocity is the *meter-per-second.*

In purely scientific work, the centimeter is almost universally adopted as the unit length, the corresponding unit velocity being the *centimeter-per-second.* (See Art. 217.)

Dimensions of unit velocity.—Since the unit velocity is so defined as to be proportional directly to the unit length and inversely to the unit time, there may be written the dimensional equation

$$\mathbf{V} = \mathbf{L}/\mathbf{T},$$

if $\mathbf{V}$ denotes the unit velocity, $\mathbf{L}$ the unit length and $\mathbf{T}$ the unit time (Art. 15).

192. Numerical Value of Velocity.—The numerical value of the velocity of a particle which moves uniformly throughout a certain interval, expressed in terms of the unit above defined, is found by dividing the length of the displacement by the duration of the interval.

Thus, let x denote the abscissa of a uniformly moving particle at the time t, reckoned from some assumed instant (taken as *origin of time*). Let x_1 and x_2 be the values of x at times t_1 and t_2. Then if v denotes the velocity,

$$v = (x_2 - x_1)/(t_2 - t_1).$$

If x and t are expressed in feet and seconds respectively, the value of v given by this formula is in feet-per-second. If x is in centimeters, v is in centimeters-per-second.

193. Sign of Velocity.—The velocity will be called plus or minus according as the particle moves in the positive or in the negative direction along the line of motion.

The formula $v = (x_2 - x_1)/(t_2 - t_1)$ gives the value of the velocity in sign as well as in magnitude; for $x_2 - x_1$ will be positive if the motion has the plus direction and negative if the motion has the minus direction.

194. Variable Motion.— If a particle receives unequal displacements in equal intervals of time, its motion is *variable*. Its velocity may still be defined as its *rate of moving*, but the above method of computing the value of this rate becomes inapplicable.

Velocity must now be understood to be a quantity which has a definite value *at any instant*, but whose value continually varies. The meaning of velocity in case of variable motion is best explained by a consideration of "average velocity."

195. Average Velocity.— If a particle moves in any manner, its *average velocity* during any given interval may be defined as the velocity of a particle which, moving uniformly, receives an equal displacement in the same interval.

The formula

$$v = (x_2 - x_1)/(t_2 - t_1)$$

always gives the value of the average velocity for the interval from t_1 to t_2.

196. Approximate Value of Velocity at an Instant.— The velocity of a particle at any instant may be computed approximately by finding the average velocity for a very short interval.

Let x denote the abscissa of the particle at the instant t, and let Δx denote the increment of x in a short interval of time Δt; that is, Δx is the displacement during the time Δt. Then $\Delta x/\Delta t$ is an approximate value of the velocity at the instant t. The approximation is closer the shorter the interval Δt.

197. True Value of the Velocity at an Instant.— The true value of the velocity at the time t is the limit approached by the

approximate value as the assumed interval is taken smaller, approaching zero. But this limit is the derivative of x with respect to t, that is,

$$v = \lim [\Delta x/\Delta t] = dx/dt.$$

This formula is the mathematical expression of the definition of *velocity at an instant.*

198. Computation of Velocity When Variable.— If the position of a particle is known at every instant, the velocity at any instant may be computed from the above formula. Thus, if x is known as a function of t, dx/dt may be determined by differentiation.

Examples.

1. The position of a particle is given by the equation $x = kt^2$, k being a constant.

(*a*) Show that the velocity at any instant is given by the formula $v = 2kt$.

(*b*) Where is the particle at the instant taken as "origin of time"?

(*c*) If, 3 sec. after the "origin of time," the particle is 12 ft. from the origin from which x is measured, what is the value of the constant k?

(*d*) Assuming condition (*c*), what is the velocity when the particle is 10 ft. from the origin?

(*e*) What is the value of k if the centimeter is taken as the unit length?

(*f*) What is the velocity when the particle is 1 met. from the origin? (Express the result in centimeter-second units.)

2. The position of a particle is given by the equation $x = 2t + t^2$, x being in feet and t in seconds.

(*a*) Compute the average velocity for each of the following intervals, beginning at the instant $t = 10$: 2 sec., 1 sec., 0.5 sec., 0.1 sec., 0.01 sec., 0.001 sec.

(*b*) Compute the exact value of the velocity at the instant $t = 10$.

Ans. (*a*) The values of the average velocity in ft.-per-sec. are 24, 23, 22.5, 22.1, 22.01, 22.001. (*b*) 22 ft.-per-sec.

3. The position of a body falling freely from rest is given approximately by the equation $x = 16.1t^2$, in which x ft. is the distance fallen in t sec. Compare the true velocity at the end of 2 sec. with the average velocity for an interval of 0.1 sec. after the instant $t = 2$.

Ans. True vel. = 64.4 ft.-per-sec. Average vel. = 66.01 ft.-per-sec.

199. Representation of Velocities by Lines.— If P (Fig. 102) represents the position of a particle at any time t and O is the as-

sumed origin in the line of motion, OP represents x, the abscissa of the particle at the time t. On another line parallel to OP, take an origin O', and lay off $O'P'$ equal (on any convenient scale) to v, the velocity of the particle at the time t. If the velocity is constant, the point P' remains fixed in position; if the velocity of P varies, P' moves. If the motion of P' is known, the velocity at every instant is known.

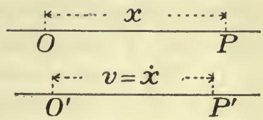


FIG. 102.

200. Notation for Time-Derivatives.— When a variable quantity is a function of the time, its derivatives are often designated by the use of dots, as follows:

$$dx/dt = \dot{x}; \quad d^2x/dt^2 = d\dot{x}/dt = \ddot{x}; \quad \text{etc.}$$

This notation will often be used in the following pages.

§ 2. *Velocity-Increment and Acceleration.*

201. Increment of Velocity.— Let v_1 and v_2 be the values of the velocity at instants t_1 and t_2 respectively; then $v_2 - v_1$ is the *velocity-increment* for the interval from t_1 to t_2. This increment may be either positive or negative, according as v_2 is (algebraically) greater or less than v_1.

Geometrical representation.—If, as described in Art. 199, the velocity at every instant be represented in magnitude and direction by a length $O'P'$ laid off from a fixed point O' (Fig. 102), the point P' will move as the velocity varies. When v increases (algebraically), P' moves in the plus direction; when v decreases, P' moves in the minus direction. The displacement of P' in any interval represents the velocity-increment.

The velocity-increment must not be confounded with the displacement of the particle. There is not necessarily any relation between them. The point P' may or may not move in the same direction as the particle.

202. Uniformly Varying Velocity.— If the velocity receives equal increments in all equal intervals of time, however small these may be, the velocity is *uniformly variable.*

Recurring to the geometrical representation given above (Fig. 102), if the velocity of P changes at a uniform rate, the point P'

moves uniformly. The motion of P' has the plus or the minus direction according as the velocity of P is (algebraically) increasing or decreasing.

203. Acceleration.— The *rate of change of the velocity* of a particle is called its *acceleration.*

If, throughout a certain interval of time, the velocity changes at a uniform rate, the acceleration is proportional directly to the total velocity-increment and inversely to the duration of the interval.

The numerical value of the acceleration depends upon the unit in which it is expressed.

204. Unit of Acceleration.—Although the unit of acceleration may be chosen arbitrarily, it is convenient to make it depend upon the units of length and time. Hence the unit acceleration is defined as that of a particle whose velocity increases by one unit in every unit of time.

Thus, with the foot and second as fundamental units, the unit acceleration is that of a particle whose velocity increases by one foot-per-second in each second.

With the meter replacing the foot as the unit length, the unit acceleration is that of a particle whose velocity increases by one meter-per-second in each second.

Similarly, a third unit of acceleration is derived from the centimeter and second as fundamental units.

Dimensions of unit acceleration.—The above definition of the unit acceleration makes it proportional directly to the unit velocity and inversely to the unit time. Representing the unit acceleration by **A**, there may be written the dimensional equation

$$\mathbf{A} = \mathbf{V}/\mathbf{T} = \mathbf{L}/\mathbf{T}^2.$$

205. Numerical Value of Acceleration.—When the velocity of a particle increases at a uniform rate during any interval, the value of the acceleration in terms of the unit above defined is computed by dividing the increment of velocity by the duration of the interval. Thus, let v denote the velocity at any instant t, v_1 and v_2 the values of v at the instants t_1 and t_2, and p the acceleration. Then

$$p = (v_2 - v_1)/(t_2 - t_1).$$

This value of p may be in feet-per-second-per-second, in meters-per-second-per-second, in centimeters-per-second-per-second, or in other units, according to the units in which t and v are expressed.

206. Algebraic Sign of Acceleration.— If $v_2 - v_1$ is negative, the value of p is negative. This means that the velocity is decreasing algebraically; that is, that the velocity-increment received by the particle in any interval has the minus direction. If p is plus, the velocity is increasing algebraically, although numerically it may be decreasing. Thus,

if v is	positive	and increasing	in magnitude,	p is	positive;
" v "	negative	" decreasing "	"	p "	" ;
" v "	positive	" decreasing "	"	p "	negative;
" v "	negative	" increasing "	"	p "	" .

207. Non-Uniform Variation of Velocity.— In case the increments of velocity in equal intervals of time are unequal, acceleration may still be defined as the rate of change of the velocity, but the above method of computing its value becomes inapplicable.

Acceleration must now be understood to be a variable quantity, having a definite value *at any instant.* Its meaning is best understood by a consideration of "average acceleration."

208. Average Acceleration.— The *average acceleration* of a particle for an interval during which its velocity varies in any way is an acceleration which, if constant, would result in the same velocity-increment in the same interval.

The formula

$$p = (v_2 - v_1)/(t_2 - t_1)$$

always gives the value of the average acceleration for the interval from t_1 to t_2.

209. Approximate Value of Acceleration at an Instant.— An approximate value of the acceleration at an instant may be found by computing the average acceleration for a very short interval of time.

Let v denote the velocity at an instant t, and Δv the velocity-increment for a short time Δt. Then $\Delta v/\Delta t$ is an approximate value of the acceleration at the instant t. The approximation is closer the shorter the interval Δt.

210. Exact Value of Acceleration at an Instant.— The exact value of the acceleration at the time t is the limit approached by the approximate value as Δt is taken smaller and smaller, approaching zero. This limit is the derivative of v with respect to t; that is,

$$p = \lim [\Delta v/\Delta t] = dv/dt.$$

Since $v = dx/dt = \dot{x}$, we have

$$p = d^2x/dt^2 = d\dot{x}/dt = \ddot{x}.$$

211. Application of Formulas for Velocity and Acceleration.—The formulas

$$v = dx/dt, \quad p = dv/dt = d^2x/dt^2,$$

are of fundamental importance in the solution of various problems in rectilinear motion. Suppose, for example, that x is known as a function of t; that is, that the position at every instant is known. In this case v and p can be determined by differentiation. Thus, if $x = f(t)$, we have

$$v = dx/dt = df(t)/dt = f'(t),$$

$$p = dv/dt = d^2x/dt^2 = df'(t)/dt = f''(t).$$

As another case, suppose the velocity or acceleration known at every instant; that is, let v or p be a known function of t. In this case the value of x in terms of t can be found by integration.

In general, if a relation is known between any or all of the quantities x, t, v and p, the motion can be completely determined either by differentiation or by integration. If the solution involves integration, either one or two arbitrary constants will be brought in, the values of which cannot be determined without additional data as to the conditions of the motion.

Thus, if there is given such a relation as

$$f(x, t, v, p) = 0,$$

the substitution of the general values of v and p as derivatives with respect to t gives a differential equation whose solution determines the motion.

Examples.

1. The position of a particle is given by the equation $x = kt^2$, k being a constant. Show that the acceleration is constant, and determine its value in terms of k.

2. In Ex. 1, if the velocity is 8 ft.-per-sec. at the instant $t = 3$, what are the values of k and of the acceleration?

3. If the position of a particle is given by the equation $x = kt^2 + k't + k''$, show that the acceleration has the same value as if the equation were $x = kt^2$.

4. The position of a particle is given by the equation $x = 10 + 5t + 2t^2 + t^3$, x being in feet and t in seconds. Compute the average

acceleration for 0.1 sec. after the instant $t = 2$, and compare it with the exact value of the acceleration at the beginning of the interval.

The velocity is given by the formula

$$v = \dot{x} = 5 + 4t + 3t^2.$$

When $t = 2$, $v = 25$. When $t = 2.1$, $v = 26.63$. Hence the average acceleration is

$$\Delta v/\Delta t = (26.63 - 25)/0.1 = 16.3 \text{ ft.-per-sec.-per-sec.}$$

The true acceleration at any instant is

$$p = dv/dt = \ddot{x} = 4 + 6t.$$

When $t = 2$, $p = 16$ ft.-per-sec.-per-sec.

5. If the velocity of a particle is given by the equation $v = 12 - 16t$, determine the position at any time.

The equation may be written

$$\dot{x} = dx/dt = 12 - 16t.$$

Integrating with respect to t,

$$x = 12t - 8t^2 + C.$$

In order that the constant of integration C may be determined, the position of the particle at some given instant must be known. Thus, if t is reckoned from the instant at which the particle is at the origin, the values $(x = 0, t = 0)$ must satisfy the equation. This requires that $C = 0$. But if the "origin of time" (*i. e.*, the instant from which t is reckoned) is chosen as the instant when $x = 4$, the values $(x = 4, t = 0)$ must satisfy the equation, and therefore $C = 4$. If x is in feet, in what units is C?

6. If $v = 12 - 16t$, determine the acceleration at the instant $t = 5$, and the average acceleration for 0.1 sec. after $t = 5$.

7. Let the velocity be given by the equation $v = 12 + 18t^2$, t being in seconds and v in centimeters-per-second. Determine the position, the velocity and the acceleration at the instant $t = 10$ sec. In order that the value of x may be completely determined, what additional data must be given?

8. With data of Ex. 7, determine the average velocity and the average acceleration during 0.5 sec. and during 0.1 sec. following the instant $t = 10$ sec.

9. The velocity of a particle at a certain instant is 100 ft.-per-sec. in the positive direction. The acceleration is constant and equal to 24 ft.-per-sec.-per-sec. in the negative direction. (*a*) When will the particle be at rest? (*b*) What will be its velocity 4 sec. later?

Ans. (*a*) At the end of $4\frac{1}{6}$ sec. (*b*) -96 ft.-per-sec.

10. The acceleration being constant, determine the position and velocity at any time.

Let f be the given constant value of p. Then the problem is to solve the differential equation

$$d^2x/dt^2 = dv/dt = f.$$

Integrating with respect to t,

$$v = dx/dt = ft + C,$$

C being a constant of integration. To determine its value, an "initial condition" must be known; that is, the value of the velocity at some definite instant must be known. Let v_0 denote the value of v when $t = 0$; then, since the last equation is true for any simultaneous values of v and t,

$$v_0 = f \cdot 0 + C; \quad \text{or} \quad C = v_0.$$

Hence $$v = \dot{x} = ft + v_0.$$

Integrating the last equation with respect to t,

$$x = \tfrac{1}{2}ft^2 + v_0 t + C'.$$

To determine the constant of integration, C', the position at some given instant must be known. Let x_0 denote the value of x when $t = 0$; then, the last equation being true for $x = x_0$ and $t = 0$, C' must equal x_0. Hence

$$x = \tfrac{1}{2}ft^2 + v_0 t + x_0.$$

11. The velocity of a falling body is observed to increase by 32.2 ft.-per-sec. during every second of its motion. How far will it fall from its position of rest in t seconds?

[The acceleration is constant and equal to 32.2 ft.-per-sec.-per-sec. Hence this is a special case of Ex. 10.]

12. The acceleration of a particle increases in direct ratio with the time reckoned from a given instant. Determine the position and velocity at any time. Let distance be expressed in centimeters and time in seconds, and assume that the acceleration increases by 4 units in 1 sec. Assume further that the velocity is zero when the acceleration is zero, and that x is reckoned from the position of the particle when the acceleration is zero. That is, the four quantities p, v, x, t are all zero together.

Ans. $x = \frac{2}{3}t^3$, $v = 2t^2$, $p = 4t$.

13. With data as in Ex. 12, where was the particle and what was its velocity 3 sec. before the instant at which the acceleration was zero?

14. If x were reckoned from the position of the particle 3 sec. before the acceleration was zero, how would the results of Ex. 12 be changed? (Make no change in the origin of time.) Does this change in the origin of abscissas imply a different case of motion?

§ 3. *Motion and Force.*

212. Laws of Motion.— In the foregoing analysis of the motion of a particle, nothing has been said of the laws in accordance with which the motions of bodies actually occur.

By studying the motions of bodies in nature, it is found that the motion of any given body is influenced by its relation to other bodies. This is often expressed by saying that one body is "acted upon" by other bodies. Such an "action" of one body upon another, measured in a particular way, is called a *force*. (Art. 32.)

Newton's three laws of motion give a concise statement of the way in which the motion of a body is influenced by forces,—*i. e.*, by the action of other bodies. A full understanding of these laws requires a more general analysis of motion than has been given in this Chapter. The present discussion of the laws of motion must be limited to the case of motion in a straight line. A formal statement of Newton's laws, with a fuller explanation of their meaning, will be given in Chapter XIV.

213. Motion of a Body Which Is Not Influenced by Other Bodies.— Newton's first law asserts that a body not acted upon by force (that is, wholly uninfluenced by other bodies) will remain at rest, or else will move in a straight line with uniform velocity.

The meaning of this law has already been briefly explained (Art. 31).

214. Effect of Constant Force on Body Initially at Rest.— If a force of constant magnitude and direction acts, for a certain interval of time, upon a body initially at rest, the body will have at the end of the interval a velocity whose direction is that of the force, and whose magnitude is proportional directly to the force and to the duration of the interval, and inversely to the mass of the body.

Let P denote the magnitude of the force, t the duration of the interval, and m the mass of the body acted upon; then at the end of the interval the velocity of the body is proportional directly to P and to t, and inversely to m; that is, it is proportional to Pt/m.

Thus, if a force P', acting for a time t' upon a particle of mass m' initially at rest, gives it a velocity v'; and if a force P'' acting for a time t'' upon a particle of mass m'' initially at rest gives it a velocity v''; then

$$v' : v'' = P't'/m' : P''t''/m''.$$

Examples.

1. A given force acting upon a mass of 1 lb. for 1 sec. gives it a velocity of 10 ft.-per-sec. What velocity would an equal force impart to a mass of 5 lbs. in 4 min.?

2. A force of magnitude P acting for 4 sec. upon a body of mass 20 lbs. gives it a velocity of 10 ft.-per-sec. What velocity will be imparted to a body of 50 lbs. mass in 9 sec. by a force of magnitude $3P$?

3. A force of 8 lbs. (*i. e.*, equal to the weight of 8 lbs. mass; see Art. 47), acting upon a body of mass m lbs. for 12 sec., gives it a velocity of 45 ft.-per-sec. What velocity will be imparted to a body of mass $7m$ lbs. by a force of 22 lbs. acting for 9 sec.?

4. Two bodies whose masses are 2 lbs. and 5 lbs., starting from rest at the same instant, are observed to have equal velocities at every subsequent instant. Compare the magnitudes of the forces acting upon them.

5. A body near the surface of the earth will, if unsupported, fall toward the earth. It is observed that two bodies of different masses, starting from rest, acquire equal velocities in equal times. What inference can be drawn as to the forces acting upon them?

6. A body falling from rest under the attraction of the earth is observed to have a velocity of 32.2 ft.-per-sec. at the end of the first second. Assuming the attraction of the earth to be a constant force, what velocity will the body have at the end of 20 sec.?

7. If g ft.-per-sec. is the velocity acquired by a falling body in 1 sec., what is its velocity at the end of t sec., assuming that the only force acting upon the body is the constant attraction of the earth?

215. Effect of Constant Force on Body Not Initially at Rest.— If a body moving in a straight line is acted upon by a force of constant magnitude whose direction coincides with that in which the body is moving, its velocity will receive, during any interval of time, an increment proportional directly to the force and to the duration of the interval, and inversely to the mass of the body.

Let m be the mass of the body, P the magnitude of the force, and let the velocity have values v_1 and v_2 at any two instants t_1, t_2. Then

$$v_2 - v_1 \text{ is proportional to } P(t_2 - t_1)/m.$$

This case obviously includes that given in the preceding Article; for if the initial velocity is zero, the final velocity is equal to the velocity-increment.

If a body is acted upon by a constant force whose direction is opposite to that in which the body is moving, the velocity is de-

creased, during any interval of time, by an amount proportional directly to the force and to the interval, and inversely to the mass. If the interval be long enough, the velocity will be reduced to zero, and will then be reversed in direction. This case and the preceding are both included in the general statement that

A constant force acting upon any body gives it a velocity-increment whose direction is that of the force, and whose magnitude is proportional directly to the force and to the time during which it acts, and inversely to the mass of the body.

If the two directions along the line of motion are distinguished as plus and minus, the sign of the velocity-increment agrees with that of the force.

It should be observed that this principle has no reference to the distance described by the body during the interval in question. It gives us a rule for estimating the *change* in the velocity; the result is independent of the actual velocity at the beginning of the interval or at any other instant. The distance passed over depends upon the value of the velocity at every instant throughout the interval, not merely upon the amount by which the velocity changes.

The above proposition is a statement of Newton's second law of motion, as applied to the motion of a body under the action of a single force whose line of action coincides with the line of motion. Newton's third law is the law of action and reaction (Art. 35). For a formal statement of the three laws, see Art. 259.

216. Equation of Motion for Particle Acted Upon by Constant Force.— Let a particle of mass m, acted upon for a time Δt by a force P, receive a velocity-increment Δv; and let a particle of mass m', acted upon by a force P', receive in a time $\Delta t'$ a velocity-increment $\Delta v'$. Then

$$\Delta v : \Delta v' = P\Delta t/m : P'\Delta t'/m';$$

or

$$m\Delta v/P\Delta t = m'\Delta v'/P'\Delta t'.$$

That is, the quantity $m\Delta v/P\Delta t$ has the same value whatever the particular case of motion considered; the force P and the mass m having any values whatever, and Δv being the increment of velocity received in a time Δt. There may therefore be written the equation

$$m\,\Delta v/P\Delta t = k,$$

or

$$\Delta v = k(P\Delta t/m), \qquad (1)$$

k being a constant. This may be called the *general equation of motion* for a particle acted upon by a constant force directed along the line of motion.

If p is the acceleration, the equation may be written

$$p = k(P/m), \quad . \quad . \quad . \quad . \quad (2)$$

since $p = \Delta v/\Delta t$ when the velocity varies at a uniform rate.

The numerical value of k depends upon the units employed in expressing the values of Δv, P, m and Δt. These units having been chosen, k can be determined by a single experiment. The nature of the experiment must be as follows:

A force of known magnitude (say P' units) is applied to a body of known mass (m' units) for a known time ($\Delta t'$ units), and the velocity produced is measured (call it $\Delta v'$ ft.-per-sec.). Then, since these values must satisfy equation (1),

$$k = m'\Delta v'/P'\Delta t'.$$

The units of force, mass, length and time may thus be chosen arbitrarily and a value of k determined which will make equation (1) true for all cases in which the same units are employed.

The unit velocity is here assumed to be derived from the units of length and of time as in Art. 191.

Examples.

1. Take the unit mass as a pound, the unit force as the weight of a pound-mass (*i. e.*, a pound-force), the unit time as the second and the unit length as the foot. Determine the value of k.

Let a body of m pounds-mass fall freely from rest under the action of gravity. Experiment shows that the velocity increases at a uniform rate. In one second the increment of velocity is about 32.2 ft.-per-sec. The force producing this effect is the weight of m pounds-mass, or in terms of the pound-force its value is m. Hence in the above value of k we may substitute $m' = m$, $P' = m$, $\Delta t' = 1$, $\Delta v' = 32.2$;

$$\therefore \quad k = (m \times 32.2)/(m \times 1) = 32.2.$$

The equation of motion for this system of units is therefore

$$p = \Delta v/\Delta t = 32.2P/m.$$

The value 32.2 ft.-per-sec.-per-sec. is only an approximate value of the acceleration of a body falling freely under gravity. The true value varies somewhat with the position on the earth's surface. If g denotes the value at any given locality, $k = g$, and the equation of motion is

$$p = g(P/m). \quad . \quad . \quad . \quad . \quad (3)$$

Since the pound-force varies with the locality in exactly the same ratio as the value of g, the numerical value of any given force (represented by P in the equation) varies inversely as g, so that equation (3) is always true if g is given its true value for the particular locality at which the pound-force is determined.

2. If the unit mass is equal to 200 lbs., the unit force equal to the weight of 10 lbs., the unit time the second, and the unit length the foot, determine the value of k. *Ans.* $k = g/20$.

3. If the pound is the unit mass, the foot the unit length, and the second the unit time, what must be the unit force in order that k may equal 1?

Ans. A force equal to $1/g$ times the weight of 1 lb.

4. If the pound-force is the unit force, the foot the unit length, and the second the unit time, what must be the unit mass in order that k may equal 1? *Ans.* A mass equal to g lbs.

5. Show that, whatever units are employed, the constant k is numerically equal to the acceleration due to the unit force acting upon the unit mass. Answer examples 1, 2, 3 and 4 by the direct application of this general result.

6. The equation of motion may be written $P = k'mp$, in which $k' = 1/k$. Show that the constant k' is numerically equal to the force which will give the unit mass the unit acceleration.

217. Kinetic Systems of Units.— It has been seen that, in the general equation of motion given above (Art. 216), the four quantities, force, mass, velocity and time (or force, mass, length and time) may be expressed in any arbitrary units, provided the value of k is properly determined. It is also apparent that k may be given any desired value by properly choosing the units of force, mass, length and time.

In order to simplify the equation of motion, let $k = 1$, and consider what restriction is thus imposed upon the choice of units.

The equation of motion becomes

$$p = \Delta v/\Delta t = P/m. \qquad . \quad . \quad . \quad (1)$$

In order to satisfy this equation it is necessary to express force, mass, length and time in such units that *a unit force acting for a unit time upon a unit mass will give it a unit velocity*.

Any system of units satisfying this requirement may be called a *kinetic system*.

It is obvious that, even with this restriction, any three of the four units named may be chosen arbitrarily. But when three are chosen as fundamental, the fourth becomes a derived unit.

For the purposes of pure science the common practice is to take as fundamental the units of *mass*, *length* and *time;* the unit force being derived from these in accordance with the requirement above stated. Two such systems of units may be mentioned.

The centimeter-gram-second system.—In this system the centimeter is the unit length, the gram the unit mass and the second the unit time. It is briefly called the C. G. S. system. The unit force is called a *dyne.*

A dyne is a force which, acting for one second upon a mass of one gram, gives it a velocity of one centimeter-per-second.

The C. G. S. system is almost universally employed in purely scientific work.

The foot-pound-second system.—In this system (called briefly the F. P. S. system) the foot is the unit length, the pound the unit mass and the second the unit time. The unit force is called a *poundal.*

A poundal is a force which, acting for one second upon a mass of one pound, gives it a velocity of one foot-per-second.

Examples.

1. A body whose mass is 40 lbs. is acted upon by a constant force of 12 poundals. What is the velocity after 4 sec. (*a*) if initially at rest, (*b*) if the initial velocity is 20 ft.-per-sec. in the direction of the force, (*c*) if the initial velocity is 20 ft.-per-sec. in the direction opposite to that of the force.

Ans. (*a*) 1.2 ft.-per-sec. (*b*) 21.2 ft.-per-sec. (*c*) 18.8 ft.-per-sec.

2. A body of 20 lbs. mass is acted upon by a constant force which in 12 sec. gives it a velocity of 60 ft.-per-sec. What is the magnitude of the force in poundals?

3. A body of 6 lbs. mass, starting from rest and falling freely under the earth's attraction, is observed to have, after 2 sec., a velocity of 64.4 ft.-per-sec. If the earth's attraction upon the body is a constant force, what is its magnitude in poundals?

4. A body of m lbs. mass, falling vertically under the action of its own weight, receives during each second a velocity of g ft.-per-sec. What is its weight in poundals? (That is, what attractive force is exerted upon it by the earth?) *Ans.* mg poundals.

5. What is the ratio between a force of one pound and a force of one poundal?

6. A body of 20 gr. mass is acted upon by a constant force of 5 dynes. Determine its velocity after 16 sec. (*a*) if initially at rest,

(*b*) if its initial velocity is 16 c.m.-per-sec. in the direction of the force, (*c*) if its initial velocity is 16 c.m.-per-sec. in the direction opposite to that of the force. *Ans.* (*c*) 12 c.m.-per-sec.

7. A body whose mass is 3 kilogr. is acted upon by a constant force which, in 5 sec., changes its velocity from 10 met.-per-sec. in one direction to 12 met.-per-sec. in the opposite direction. What is the value of the force?

Ans. 1,320,000 dynes in the direction of the final velocity.

8. A body of 2,000 gr. mass, starting from rest and falling freely under the earth's attraction, has at the end of 2 sec. a velocity of 19.6 met.-per-sec. If the earth's attraction upon the body is a constant force, what is its magnitude in dynes?

Ans. 1.96×10^6 dynes.

218. Engineers' Kinetic System.—For the purposes of the engineer the most convenient unit of force is the weight of a definite mass at the earth's surface. Such a unit, though not exactly the same for all localities, is sufficiently definite for most practical purposes.

Let the pound-force (already defined as the weight of a pound-mass) be taken as the unit force, let the foot and second be taken as units of length and time, and let the unit mass be so determined as to satisfy the equation $P = m(\Delta v/\Delta t) = mp$.

The unit mass must now be defined as *a mass whose velocity will increase by one foot-per-second during each second if acted upon by a force of one pound.*

The ratio of this unit to the pound-mass may readily be determined. A force of 1 lb. acting upon a mass of 1 lb. for 1 sec. gives it a velocity of g ft.-per-sec. The same force acting upon a mass of g lbs. for 1 sec. will therefore give it a velocity of 1 ft.-per-sec. The required unit mass is therefore g times as great as the pound-mass.

If this system of units is employed, a mass whose value is given in pounds must be reduced to the unit just defined by dividing by g.

It will hereafter be assumed, unless otherwise expressly stated, that a kinetic system of units is employed, so that the equation of motion takes the form

$$P = mp.$$

Examples.

1. A mass of half a ton is acted upon by a force of 50 lbs.

Write the equation of motion, using the pound as the unit force. What is the acceleration? [1 ton = 2,000 lbs.]

Ans. $p = 1.61$ ft.-per-sec.-per-sec.

2. With French units, let the unit length be the meter, the unit time the second and the unit force the weight of a kilogram. Determine the value of the unit mass in terms of the kilogram, so that the equation $P = mp$ may be satisfied.

3. A mass of 200 kilograms is acted upon by a force equal to half its weight. Write the equation of motion, taking units as in Ex. 2. Determine the acceleration.

Ans. $p = 4.9$ met.-per-sec.-per-sec.

219. Dimensions of Units in Kinetic System.—The relation which must subsist among the units in order that the constant k in the general equation of motion shall be unity may be expressed by a dimensional equation. Let the units be represented by the same symbols as heretofore (Arts. 15, 191, 204).

The equation to be satisfied is $P = mp$. The dimensional equation is therefore

$$\mathbf{F} = \mathbf{MA}.$$

But also

$$\mathbf{V} = \mathbf{L}/\mathbf{T}; \quad \mathbf{A} = \mathbf{V}/\mathbf{T} = \mathbf{L}/\mathbf{T}^2;$$

hence

$$\mathbf{F} = \mathbf{ML}/\mathbf{T}^2.$$

If, as in the C. G. S. and the F. P. S. systems (Art. 217), the fundamental units are **M, L** and **T,** this equation gives the dimensions of the unit force. If, however, **F, L** and **T** are made fundamental, the equation gives the dimensions of the unit mass; in this case it may be written

$$\mathbf{M} = \mathbf{FT}^2/\mathbf{L}.$$

In accordance with scientific usage, we shall generally regard **M, L** and **T** as fundamental units in the general equation of motion and all equations derived from it; force being regarded as having the dimensions given by the above equation.

220. Effect of Two or More Constant Forces.—If a body moving in a straight line be acted upon by two or more forces directed along the line of motion, it receives, during any interval, a velocity-increment equal to the algebraic sum of the increments which would be produced by the forces acting separately. The same effect would be produced by a single force equal to the algebraic sum of the several forces. That is, the resultant of any number of forces acting

upon the same particle and directed along its line of motion is a single force equal to their algebraic sum.

221. Effect of Variable Force.— If a body is acted upon by a force of variable magnitude, the velocity-increment produced in any given interval cannot be estimated so simply as in the case of a constant force. In this case the equation of motion becomes a differential equation.

If a body of mass m receives a velocity-increment Δv in a time Δt, the equation

$$P = m(\Delta v/\Delta t)$$

gives the magnitude of a constant force which will produce the same velocity-increment in the same time. If the interval Δt is long, the actual magnitude of the force may vary widely from this value. But if the time Δt be taken smaller and smaller, approaching the limit zero, Δv also approaching zero, the value of P given by the equation approaches as a limit the actual magnitude of the force at the beginning of the interval. Hence, if P denotes the value of the force at any instant,

$$P = \lim\ [m(\Delta v/\Delta t)] = m(dv/dt).$$

The determination of the change of velocity produced by the force in any finite time requires the integration of this equation. This is not possible unless the law of variation of P is known.

222. Equation of Motion of a Particle Acted Upon by Any Number of Constant or Variable Forces.— It is now evident that, if a kinetic system of units be employed, the equation of motion may be written

$$P = m(dv/dt), \quad \text{or} \quad P = mp,$$

if p is the instantaneous value of the acceleration and P the instantaneous value of the algebraic sum of all forces acting on the particle.

If the position of the particle be specified by its distance x from a fixed point in the line of motion, $p = dv/dt = d^2x/dt^2$, and the equation of motion may be written

$$P = m(d^2x/dt^2).$$

CHAPTER XIII.

MOTION IN A STRAIGHT LINE: APPLICATIONS.

§ 1. *General Method.*

223. Classes of Problems.—The equation

$$P = mp \quad . \quad . \quad . \quad . \quad (1)$$

may always be applied in the solution of problems relating to the motion of a particle in a straight line. The problems that may conceivably arise are of various kinds, depending upon what is known and what unknown regarding the motion. The most important cases are the following:

(1) The motion being known, it is required to determine the resultant force.

(2) The forces being known (as functions of the position or of the time or of both), it is required to determine the motion.

The second of these problems will require the solution of a differential equation, and may be called the *inverse* problem; the first will be called the *direct* problem, since it involves no integration.

224. Direct Problem: To Determine the Resultant Force When the Motion Is Known.—If the acceleration is known at every instant, the resultant force can be determined by substitution in the equation $P = mp$. If the position or the velocity is known as a function of the time, the acceleration can be found by differentiation.

Examples.

1. A body of 1 lb. mass, starting from rest, moves so that its distance from the starting point at every instant is given by the formula $x = 16.1t^2$, x being in feet and t in seconds. Required the magnitude of the resultant force acting on the body at any instant.

Ans. The force is constant and equal to 32.2 poundals.

2. If the velocity of a body is constant, what is the magnitude of the resultant force acting on it?

3. If the position of a particle of mass m is given by the formula $x = a \sin bt$, determine the value of the resultant force acting upon it as a function of x. *Ans.* $P = -mb^2x$.

4. A body of m lbs. mass, acted upon by no force except the earth's attraction, is observed to receive each second a velocity-increment of g ft.-per-sec. What is the magnitude of the force acting upon it?

Ans. mg poundals.

5. A body whose mass is 18 lbs. moves so that its position at any instant is given by the equation $x = 5t^2 + 6t + 8$, t being in seconds and x in feet. Required the magnitude of the resultant force acting upon it at any instant.

Ans. 180 poundals or 5.59 pounds-force.

6. What force will give an acceleration of 1,000 c.m.-per-sec.-per.-sec. to a mass of 600 gr.?

7. What constant force acting upon a particle of m grams mass will increase its velocity by g c.m.-per-sec. in 1 sec.?

8. What is the weight in dynes of a mass of m grams?

Ans. mg, if g is the acceleration due to gravity, expressed in c.m.-per-sec.-per-sec. Its value is known to be about 981.

9. The velocity of a particle is proportional to its distance from a fixed point, and is 24 ft.-per-sec. when the distance from the fixed point is 2 ft. If the mass of the particle is 4 lbs., what is the value of the resultant force acting upon it when 8 ft. from the fixed point? Also when 3 ft. from the fixed point?

Ans. 4,608 poundals. 1,728 poundals.

225. Inverse Problem: To Determine the Motion When the Forces Are Known.— If every force is known as a function of one or more of the three quantities x, t, v, the general equation $P = mp$ becomes a differential equation, the complete solution of which determines x and v as functions of t.

In general two integrations will be required, thus introducing two constants. To determine the constants, some information must be given concerning the motion at one or more definite instants, or in some one or more definite positions. The information usually available is the following: The velocity and position at a certain instant are completely known.

Note on constants of integration.—The method of determining constants of integration has already been illustrated in several particular cases. A clear understanding of the general method is of importance to the student. To determine such a constant it is always necessary to have some information in addition to that which enables us to write the differential equation.

Let there be given any equation containing two variables, x and y, and a constant, C. Then the value of C can be determined if one

pair of simultaneous values of x and y be known; that is, if it be known that "when $x =$ some known value, $y =$ some known value." Thus, let the given equation be

$$f(x, y, C) = 0.$$

If it is known that when $x = a$, $y = b$, a and b being known constants, there may be written

$$f(a, b, C) = 0,$$

from which C can be determined.

In a problem relating to the motion of a particle, the variables being usually some two of the quantities x, t and v, the information necessary for determining the constant is equivalent to some knowledge as to the position at some instant, or as to the condition of motion of the particle either at some instant or in some position.

It is to be noticed that the same method will serve for determining any constant in an equation, whether introduced by integration or otherwise.

§ 2. *Motion Under Constant Force.*

226. Solution of General Problem.— Let a particle of mass m be acted upon by forces whose resultant has the constant value P directed parallel to the line of motion. Let O (Fig. 103) be a fixed point in the line of motion, and let x be the distance of the particle from O at the time t. The equation of motion is then

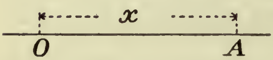


FIG. 103.

$$m(d^2x/dt^2) = P = \text{constant.} \qquad (1)$$

Two integrations give

$$m(dx/dt) = Pt + C_1; \qquad (2)$$

$$mx = \tfrac{1}{2}Pt^2 + C_1 t + C_2. \qquad (3)$$

To detemine C_1 and C_2, let it be known that at a certain instant t_0 the velocity is v_0 and the abscissa of the particle x_0. Then from (2),

$$C_1 = mv_0 - Pt_0;$$

and from (3),

$$C_2 = mx_0 - \tfrac{1}{2}Pt_0^2 - (mv_0 - Pt_0)t_0 = mx_0 - mv_0 t_0 + \tfrac{1}{2}Pt_0^2.$$

Since the origin of time may be chosen arbitrarily, let t be reckoned from the instant at which $v = v_0$ and $x = x_0$; then $t_0 = 0$, and

$$C_1 = mv_0; \quad C_2 = mx_0.$$

Equations (2) and (3) now become

$$m(dx/dt) = Pt + mv_0; \quad \text{or} \quad m(v - v_0) = Pt; \quad . \quad (4)$$

$$mx = \tfrac{1}{2}Pt^2 + mv_0 t + mx_0. \quad . \quad . \quad . \quad (5)$$

These equations give the velocity and position at any instant.

227. Motion of a Body Falling Vertically Near the Earth.— A body near the surface of the earth is attracted toward the earth's center with a force which varies as the body moves upward or downward. If the range of motion is small compared with the earth's radius, the force varies so little that for most purposes it may be regarded as constant. Hence if a body starts from rest, or is projected vertically upward or downward, and is then left to the influence of the earth's attraction, it presents a case of rectilinear motion under a constant force.

Experience shows that bodies of unequal mass, acted upon by gravity alone, are equally accelerated. Their weights are therefore proportional to their masses. Let W_1 and W_2 denote the weights of two bodies whose masses are m_1 and m_2; then

$$W_1/W_2 = m_1/m_2.$$

This equation is true for any units of force and mass. If these units satisfy the condition prescribed in Art. 217 (so that the unit force gives the unit mass the unit acceleration), the acceleration of the mass m_1 acted upon by the force W_1 is W_1/m_1, and the acceleration of the mass m_2 acted upon by the force W_2 is W_2/m_2. If the known value* of the acceleration due to gravity is denoted by g,

---

*The value of g varies with the position on the earth. This variation is approximately represented by the following formula: Let φ denote the latitude and h (centimeters) the elevation above sea-level. Then in C. G. S. units (c.m.-per-sec.-per-sec.),

$$g = 980.6056 - 2.5028 \cos 2\varphi - 0.000003h.$$

In the numerical exercises in this book in which British units are employed the value 32.2 ft.-per-sec.-per-sec. may be used; in French units the value 981 c.m.-per-sec.-per-sec. is sufficiently correct.

The effect of the earth's rotation on the value of g is considered in Art. 311.

$$W_1/m_1 = W_2/m_2 = g.$$

Or, if W is the weight of any body of mass m,

$$W/m = g.$$

This equation may also be written

$$W = mg, \quad \text{or} \quad m = W/g.$$

Thus, in any equation in which kinetic units are employed, the weight of a body of mass m may be put equal to mg.

The equation of motion

$$P = mp,$$

applied to the case of a particle of mass m acted upon by no force except its weight, becomes

$$mg = mp, \quad \text{or} \quad p = g.$$

This may be written

$$d^2x/dt^2 = g = \text{constant}, \quad . \quad . \quad . \quad (1)$$

if x denotes the distance of the particle from a fixed point in the line of motion. The positive direction for x is downward.

Let equation (1) be integrated, and let the conditions for determining the constants of integration be that when $t = 0$, $x = x_0$ and $v = v_0$. The first integration gives

$$dx/dt = gt + v_0; \quad . \quad . \quad . \quad (2)$$

and the second gives

$$x = \tfrac{1}{2}gt^2 + v_0 t + x_0. \quad . \quad . \quad . \quad (3)$$

These results might have been deduced immediately from equations (4) and (5), Art. 226, by substituting g for P/m.

Consider the three cases in which the initial velocity is zero, positive, and negative, respectively.

(1) Let the body fall from rest, and take the starting point as the origin for reckoning x, t being reckoned from the instant when the body is at the origin. Then $v_0 = 0$, and equations (2) and (3) become

$$dx/dt = v = gt; \quad . \quad . \quad . \quad . \quad (4)$$

$$x = \tfrac{1}{2}gt^2. \quad . \quad . \quad . \quad . \quad (5)$$

(2) If the body has initially a velocity downward, the general formulas (2) and (3) apply, v_0 having a positive value. Let $v_0 = V$, and let x be measured from the point of projection, so that $x_0 = 0$.

Then $$dx/dt = v = gt + V; \qquad (6)$$

$$x = \tfrac{1}{2}gt^2 + Vt. \qquad (7)$$

(3) If the body is given an initial velocity V upward, and if x is reckoned from the initial position and is positive downward, $v_0 = -V$, $x_0 = 0$, and equations (2) and (3) become

$$dx/dt = v = gt - V; \qquad (8)$$

$$x = \tfrac{1}{2}gt^2 - Vt. \qquad (9)$$

EXAMPLES.

1. Prove that the velocity v of a body which has fallen vertically a distance x from rest is given by the formula $v^2 = 2gx$.

2. Taking the value of g in centimeter-second units as 981, compute the velocity of a body after falling 10 met. from rest.

3. A body is projected upward with a velocity of 100 ft.-per-sec. When will it come to rest, how high will it rise, and when will it return to the starting point?

4. A body is projected upward with a velocity of 80 ft.-per-sec. After what time will it be 20 ft. above the initial position? Explain the double answer. Take $g = 32.2$ ft.-sec. units.
Ans. After 0.26 sec. or 4.69 sec.

5. A body is projected upward with velocity V. Show that it will rise to a height $V^2/2g$; that it will come to rest after a time V/g; and that it will return to the point of projection after a time $2V/g$.

6. A body is dropped into a well 84 ft. deep. How long before the sound of striking the bottom will be heard, if the velocity of sound is 1,100 ft.-per-sec.? *Ans.* 2.36 sec.

7. A body is dropped into a well, and the sound of striking the bottom is heard after 4 sec. How deep is the well? *Ans.* 231 ft.

8. A body is projected upward with velocity V. Show that after rising a distance h its velocity is given by the formula $v^2 = V^2 - 2gh$.

9. If a body is moving vertically under the action of gravity, prove that its average velocity during any interval of time is equal to its velocity at the middle instant of the interval. The same is true in any case of constant acceleration.

10. At a certain instant a body (acted upon by gravity alone) is moving upward at the rate of 10 ft.-per-sec. What is its average velocity for the next half-second? Determine its final position by means of the average velocity.

11. What is the average velocity of a falling body during the nth second after starting from rest? *Ans.* $(n - \tfrac{1}{2})g$ ft.-per-sec.

12. What distance (in centimeters) is described by a falling body during the 5th second after starting from rest?

13. An elevator, starting from rest, has a downward acceleration $g/2$ for 1 sec., then moves uniformly for 2 sec., then has an upward acceleration $g/3$ until it comes to rest. (*a*) How far does it descend? (*b*) A person whose weight is 140 lbs. experiences what pressure from the elevator during each of the three periods of its motion?

Ans. (*a*) $13g/8$ ft. (*b*) 70 pounds-weight; 140 pounds-weight; $186\frac{2}{3}$ pounds-weight.

14. A body of m lbs. mass rests upon a horizontal platform. If the platform begins to fall with acceleration g, what pressure does it exert upon the body? What is the pressure if the platform begins to rise with acceleration g?

15. In Ex. 14, determine the pressures in the two cases in which the acceleration is $2g$ upward and $2g$ downward.

16. Equal masses of m lbs. each rest upon two platforms, one of which has at a certain instant a velocity of 20 ft.-per-sec. upward and the other a velocity of 20 ft.-per-sec. downward. Both platforms have an upward acceleration $g/4$. Compare the pressures of the platforms on the two bodies.

17. Velocity is imparted to a body of 5 lbs. mass by means of an attached string whose breaking strength is a pull of 2 lbs. How great a velocity can the body receive in 2 sec.?

Ans. $4g/5$ ft.-per-sec.

18. A string which can just sustain a mass of 4 lbs. against gravity is attached to a body whose mass is 1 lb. which rests upon a smooth horizontal plane. Is it possible to break the string by a horizontal jerk? How great an acceleration can be given to the body by means of the string?

19. A ball whose mass is 5 oz. is moving at the rate of 100 ft.-per-sec. when it receives a blow which exactly reverses its velocity. If the force exerted upon the ball is constant and acts for 0.1 sec., what is its magnitude?

Ans. 19.4 pounds-force. Actually, the force would increase from 0 up to a value much greater than 19.4 lbs. and then decrease to 0. The average force is 19.4 if the time occupied by the blow is 0.1 sec.

20. In a locality where the value of g is 32.2 ft.-per-sec.-per-sec. a body of m lbs. mass falls 15.9 ft. from rest in one sec. What is the average value of the resistance of the air?

Ans. $0.0124m$ pounds-force.

§ 3. *Force Varying With Distance From a Fixed Point.*

228. General Problem: Force Any Function of Distance.— In dealing with the the forces of nature, an important case to be considered is that in which the force acting upon a particle is directed toward or from a fixed point (or one which may be regarded as fixed), its magnitude being some function of the distance of the particle from the point. This case will now be considered, the motion being restricted to a straight line containing the fixed point.

The point toward (or from) which the force is directed may be called the *center of force*. If the force is directed toward the center it is called *attractive;* if from the center, *repulsive*. For convenience let the origin (O, Fig. 104) be taken at the center of force, and let the force be reckoned as if it were repulsive in all cases. Let P be the magnitude of the force for any position of the particle; then the general equation (Art. 222) is $m(d^2x/dt^2) = P$, or

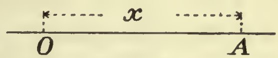


FIG. 104.

$$m(d\dot{x}/dt) = P.$$

Since in the present case P is supposed to be a function of x, the equation becomes

$$m(d\dot{x}/dt) = f(x). \quad . \quad . \quad . \quad . \quad (1)$$

In any special case the form of the function $f(x)$ is known, and the solution of the differential equation (1) gives the relation between x and t, and also the relation between $\dot{x}$ (or v) and t, thus determining the motion completely. The values of x and v will involve constants of integration, to determine which certain "initial conditions" must be specified.

In the following Articles special cases of equation (1) will be considered. One important *general* result may, however, be here noticed.

Multiplying both members of (1) by $\dot{x}\,dt = dx$, the first member becomes integrable. Thus, we have first,

$$m\dot{x}\,d\dot{x} = f(x)dx.$$

Integrating,

$$\tfrac{1}{2}m\dot{x}^2 = \int f(x)dx + C = F(x) + C,$$

$F(x)$ being obtained by immediate integration as soon as the form of $f(x)$ is known. The last equation shows that the velocity is a

13

function of the position; that is, if the particle comes more than once into the same position, the velocity has the same value, except that its direction may be reversed.

The quantity $\frac{1}{2}m\dot{x}^2$ or $\frac{1}{2}mv^2$ is an important one, and is discussed in Chapter XVII. The name *kinetic energy* is given to it, for reasons to be explained in that chapter.

229. Attractive Force Proportional Directly to the Distance.— An important case is that in which the magnitude of the force is proportional directly to the distance of the particle from the fixed point. The force may be either attractive or repulsive. Consider first the case in which it is attractive,— *i. e.*, directed always *toward* the fixed point.

Let P' be the magnitude of the attractive force when the particle is at the unit distance from O (Fig. 104); then for any distance x the force is $P = -P'x$, and the equation of motion becomes

$$m\ddot{x} = -P'x,$$

or

$$dv/dt = \ddot{x} = -P'x/m = -kx, \qquad . \qquad . \quad (1)$$

if k be written for the attractive force *per unit mass* at unit distance from O. The reason for the minus sign is that the force, being always directed toward the point O, is always opposite in sign to x.

In order to completely determine the motion, certain "initial conditions" must be known, in addition to the differential equation (1) which expresses a relation that is satisfied in every position of the particle. It will be well to state explicitly all the data which serve to determine the solution of the problem. Let the following assumptions be made:

(*a*) The origin being at O, the fixed point toward which the force is always directed, let the positive direction be toward the right, so that when the particle is on the right of O, x is positive and the force negative; and when the particle is on the left of O, x is negative and the force positive.

(*b*) Take the "origin of time" as the instant at which the particle is at the point O.

(*c*) Assume that when the particle is at a distance a from the origin (in positive direction) its velocity is zero.

To integrate (1), multiply the first member by v, and the last by its equal dx/dt; there results, $v(dv/dt) = -kx(dx/dt)$, or

$$v\,dv = -kx\,dx.$$

Multiplying through by 2 and integrating,

$$v^2 = -kx^2 + C.$$

Applying condition (*c*), we find $C = ka^2$; hence the last equation may be written

$$v^2 = k(a^2 - x^2), \quad . \quad . \quad . \quad . \quad (2)$$

which gives the velocity in any position.

To determine a relation between x and t, a second integration is required. Equation (2) may be written

$$(dx/dt)^2 = k(a^2 - x^2),$$

or $$k^{1/2}dt = dx/\sqrt{a^2 - x^2}.$$

Integrating, $$k^{1/2}t = \sin^{-1}(x/a) + C'.$$

From condition (*b*) it follows that when $t = 0$, $x = 0$, and therefore $C' = -\sin^{-1}(0) = 0$. Hence the equation becomes

$$k^{1/2}t = \sin^{-1}(x/a),$$

or $$x = a \sin(k^{1/2}t). \quad . \quad . \quad . \quad . \quad (3)$$

Equation (3) gives the value of x at any time.

[In determining the value of C', it was assumed that the angle whose sine is 0 is 0. But this is only one of many allowable values. The general value would be $\sin^{-1}(0) = n\pi$, where n is any integer. Using this value we have

$$\sin^{-1}(x/a) = k^{1/2}t + n\pi,$$

whence

$$x/a = \sin(k^{1/2}t + n\pi) = \sin(k^{1/2}t)\cos n\pi + \cos(k^{1/2}t)\sin n\pi.$$

But $\sin n\pi = 0$, and $\cos n\pi = \pm 1$; the sign depending on whether n is even or odd. Hence

$$x = \pm a \sin(k^{1/2}t).$$

The double sign shows that the assumed conditions are not sufficient to fully determine the motion; it is left uncertain whether the motion has the positive or the negative direction when $t = 0$. The former supposition will be adopted, and the plus sign used, as in equation (3).]

A discussion of equation (3) shows that x is *periodic;* that is, in successive equal intervals it passes repeatedly through the same series of values. The greatest value of x is a; and it is seen that $x = a$ as often as $\sin(k^{1/2}t) = 1$; that is, when

$$k^{1/2}t = \pi/2, \quad 5\pi/2, \quad 9\pi/2, \quad \text{etc.};$$

or when $$t = \pi/2k^{1/2}, \quad 5\pi/2k^{1/2}, \quad 9\pi/2k^{1/2}, \quad \text{etc.}$$

The interval between any two successive values of t in this series is $2\pi/k^{1/2}$; which is the interval required for the particle to return to the point at which it is at rest after once leaving it.

Again, the greatest negative value of x is $-a$; which occurs when $\sin (k^{1/2}t) = -1$; that is, when

$$k^{1/2}t = 3\pi/2, \quad 7\pi/2, \quad 11\pi/2, \quad \text{etc.};$$

or when $$t = 3\pi/2k^{1/2}, \quad 7\pi/2k^{1/2}, \quad 11\pi/2k^{1/2}, \quad \text{etc.}$$

The interval between two successive values of t in this series is $2\pi/k^{1/2}$.

It is also seen that the interval between two instants at which $x = a$ and $x = -a$ is $\pi/k^{1/2}$. Again, the interval between the instants when the particle is at O and at A is seen to be $\pi/2k^{1/2}$. Hence, if A' is the point at which $x = -a$, it is seen that, in successive intervals of $\pi/2k^{1/2}$ sec., the particle passes from A to O, from O to A', from A' to O, and from O to A; and then repeats the cycle.

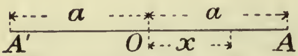


FIG. 105.

Motion in accordance with this law of force is called *harmonic motion*, and is of great importance in mathematical physics.

230. Repulsive Force Proportional Directly to the Distance.— Consider next the case in which the force is *repulsive*,— *i. e.*, acts always away from the fixed point,— and let the magnitude of the force vary in direct ratio with the distance from the center of force.

Let P' be the magnitude of the force when the particle is at the unit distance from O; then $P = P'x$, and the equation of motion is

$$m\ddot{x} = P'x,$$

or $$dv/dt = \ddot{x} = P'x/m = kx, \quad . \quad . \quad . \quad (1)$$

where k is a positive constant, and means the magnitude of the repulsive force *per unit mass* when the particle is at unit distance from O.

Equation (1) is identical with the differential equation for the motion when the force is attractive (equation (1), Art. 229), except that $-k$ takes the place of k. The integration may therefore be carried out in the same manner; and if the same initial conditions are assumed, the final result will take the same form, with the substitution of $-k$ for k. The position would be given by the equation corresponding to (3) of Art. 229:

$$x = a \sin (t\sqrt{-k}).$$

In order to avoid this imaginary form, the integration may be effected in a different manner. It will be found necessary to change the initial conditions as the solution proceeds.

Integrating as in Art. 229,

$$(dx/dt)^2 = k(x^2 - a^2), \quad . \quad . \quad . \quad (2)$$

the constant being determined from condition (c) assumed in the preceding case, that when $x = a$, $v = 0$. Equation (2) may be written

$$k^{\frac{1}{2}}dt = dx/\sqrt{x^2 - a^2}.$$

Integrating,

$$k^{\frac{1}{2}}t = \log\,(x + \sqrt{x^2 - a^2}) + C.$$

If, as in the preceding case, it be assumed that $x = 0$ when $t = 0$ (condition (b) of Art. 229), the value of C is imaginary. This shows that such a condition is inconsistent with the condition already assumed; if the particle is at rest when $x = a$, it can never pass through the origin. This is of course obvious in the case of a repulsive force; it is, in fact, evident that x can never be less than a. Therefore, instead of assuming condition (b) of Art. 229, let t be reckoned from the instant when $x = a$. This gives $C = -\log a$, and therefore

$$k^{\frac{1}{2}}t = \log\,[(x + \sqrt{x^2 - a^2})/a],$$

or

$$x + \sqrt{x^2 - a^2} = ae^{k^{\frac{1}{2}}t}.$$

Solving for x,

$$x = \tfrac{1}{2}a(e^{k^{\frac{1}{2}}t} + e^{-k^{\frac{1}{2}}t}). \quad . \quad . \quad . \quad (3)$$

Here e denotes the base of the natural system of logarithms, its value being $2.718+$.

Equation (3) shows that the motion is not periodic, as in the preceding case. If $t = 0$, $x = a$; and this is the least value x can have. Again, if t be given equal positive and negative values, the corresponding values of x are equal. Hence the motion after the instant $t = 0$ is exactly the reverse of the motion before that instant. As t increases, x increases; and the particle never returns after it begins to recede from its nearest position to O.

EXAMPLES.

1. Let the force be attractive, its magnitude at 1 ft. from the center of force being 4 poundals per pound of mass of the attracted particle; and let the particle be at rest at a certain instant at 10 ft. from the center. Determine the position and velocity in terms of t.

2. In the same case, what are the position and velocity 5 sec. after the particle is at rest?

Ans. $x = -8.39$ ft.; $v = +10.89$ ft.-per-sec.

3. In the same case, with what velocity does the particle pass the center of force? *Ans.* 20 ft.-per-sec.

4. How often does the particle return to the starting point?

Ans. At intervals of 3.1416 sec.

5. Solve examples 1 and 2, assuming the force to be repulsive, the remaining data being the same as before.

Ans. When $t = 5$, $x = 110,100$ ft., $v = 220,200$ ft.-per-sec.

6. Solve the problem of the motion of a particle under a force varying directly as the distance from a fixed point in the line of motion and directed away from that point, taking all data as in the above general solution with the following exceptions: Instead of the condition $v = 0$ when $x = a$, assume that $v = v_0$ when $x = 0$.

7. Solve with data as in Ex. 5, except that the velocity is to be 10 ft.-per-sec. when the particle passes the center of force.

8. Let the force be attractive, its magnitude at 1 met. from the center of force being 1,000 dynes per gram of mass of the attracted particle. Let the particle start from rest at 150 c.m. from the center. Determine the position and velocity at any time.

9. With data as in Ex. 8, determine the velocity when the particle is 100 c.m. from the center of force, and determine the position and velocity 10 sec. later.

Ans. When $x = 100$ c.m., $v = 354$ c.m.-per-sec.

10. Assume data as in Ex. 8, except that the force is repulsive. Determine the position and velocity 10 sec. after the particle is at rest.

231. Force Varying Inversely as the Square of the Distance.— Let a particle be attracted toward a fixed point with a force whose magnitude varies inversely as the square of the distance from that point. Let P' denote the magnitude of the force when the particle is at unit distance from the center of attraction; then the equation of motion is

$$m\ddot{x} = -P'/x^2,$$

or

$$\ddot{x} = d\dot{x}/dt = -k/x^2, \quad . \quad . \quad . \quad (1)$$

in which k ($= P'/m$) is the magnitude of the attractive force *per unit mass* at unit distance from the origin.*

---

*It is to be noticed that equation (1) applies only when the particle is on the positive side of the origin. The equation makes the acceleration negative for all values of x, while in fact the force (and therefore the acceleration) is positive when x is negative. Therefore the equation must be used for positive values of x only. For negative values of x the equation must be $\ddot{x} = k/x^2$.

Let equation (1) be integrated subject to the condition that the particle is at rest when at a distance a from the origin, and that t is reckoned from that instant. The first integration can be performed by the method described in Art. 228.

Multiplying through by dx and integrating,

$$\tfrac{1}{2}\dot{x}^2 = k/x + C.$$

The initial conditions make the constant equal to $-k/a$, hence

$$\dot{x}^2 = (dx/dt)^2 = 2k(1/x - 1/a). \qquad (2)$$

Equation (2) gives the velocity when the position is known. To find the relation between x and t, we have

$$dt\sqrt{2k/a} = dx\sqrt{x/(a-x)}.$$

Integrating

$$t\sqrt{2k/a} = -\sqrt{ax - x^2} + \tfrac{1}{2}a\sin^{-1}[(2x-a)/a] + C'.$$

If $t = 0$ when $x = a$ (as above assumed), $C' = -\tfrac{1}{2}a\sin^{-1}(1)$; hence

$$t\sqrt{2k/a} = -\sqrt{ax - x^2} + \tfrac{1}{2}a\sin^{-1}[(2x-a)/a] - \tfrac{1}{2}a\sin^{-1}(1),$$

or

$$t\sqrt{2k/a} = -\sqrt{ax - x^2} + \tfrac{1}{2}a\cos^{-1}[(2x-a)/a]. \qquad (3)$$

232. Motion Under the Attraction of the Earth.— If the earth were a sphere of uniform density throughout, or a sphere in which the density had the same value at all points equally distant from the center, its attraction upon any body outside its surface would vary inversely as the square of the distance from the center. (Art. 183.) In the actual case the attraction is very nearly expressed by the law stated. Hence equations (1), (2) and (3) of the preceding Article may be applied with small error to the motion of a body falling vertically toward the earth from a great height, supposing gravity to be the only force acting. In Art. 227 the motion of a falling body was discussed on the supposition that the attraction of the earth upon it is constant. In the present case the range of motion is supposed to be so great that a more accurate treatment is desirable.

In order to apply the results of Art. 231 to the present problem, it is necessary to determine the value of the constant k. If R denotes the radius of the earth, we know that when $x = R$ the acceleration is $-g$. Hence from equation (1) we have

$$-g = -k/R^2, \quad \text{or} \quad k = gR^2.$$

If g is in feet-per-second-per-second, R must be in feet, and k means the attraction in poundals on a pound-mass if placed one foot from the earth's center (supposing the same law of variation of the earth's attractive force to hold within the surface as without). The value of R may be taken as 20,900,000 feet.

Examples.

1. A body starts from rest at a height above the surface equal to the earth's radius. Compute the velocity when the surface is reached.

Putting $k = gR^2$, the velocity is given by equation (2) (Art. 231), which may be written

$$v^2 = 2gR^2(1/x - 1/a).$$

In this example $a = 2R$; hence for $x = R$ we have

$$v^2 = 2gR^2(1/R - 1/2R) = gR.$$

In feet-per-second,

$$v = \sqrt{32.2 \times 20{,}900{,}000} = 25{,}900.$$

2. What is the greatest velocity a body could acquire in falling from rest to the earth's surface? [Put $a = \infty$, $x = R$.]

3. With what velocity must a body be projected upward at the earth's surface in order that it may never return?

Ans. A velocity not less than 6.95 miles-per-sec.

4. Deduce a formula for the velocity acquired by a body in falling to the surface from a height h.

Putting $a = R + h$ and $x = R$, the above general formula for v^2 becomes

$$v^2 = 2gh[R/(R + h)].$$

If h is small compared with R, we may put as a first approximation

$$R/(R + h) = 1; \quad \therefore \quad v^2 = 2gh.$$

This is identical with the formula which would apply if the attraction were constant. See Ex. 1, Art. 227.

In using the accurate formula, if h/R is small we may write

$$R/(R + h) = (1 + h/R)^{-1}$$
$$= 1 - (h/R) + (h/R)^2 - (h/R)^3 + \quad . \quad . \quad . \quad ;$$

and therefore

$$v^2 = 2gh[1 - (h/R) + (h/R)^2 - (h/R)^3 + \quad . \quad . \quad . \quad].$$

By taking any number of terms of this series an approximate result may be obtained which is correct to any desired degree.

5. Let a body fall to the surface from a height of 5,000 ft. Compute the velocity acquired, using first the approximate formula and second the accurate formula. (Take $g = 32.2$.)

6. Determine the value of k, using C. G. S. units.

§ 4. *Miscellaneous Problems.*

233. Constant Force Combined With Central Force.— Let a body be acted upon simultaneously by a constant force and a force directed toward a fixed point. Then if P_1 and P_2 are the values of the two forces at any instant, the general equation becomes

$$m\ddot{x} = P_1 + P_2, \quad . \quad . \quad . \quad . \quad (1)$$

and we must put for P_1 and P_2 their values as functions of x and t. If the origin is taken at the center of force, and the central force is a function of the distance from the center, we have

$$P_1 = \text{constant}; \quad P_2 = f(x).$$

The only case that will be discussed is the following:

Body suspended by elastic string.— A body is suspended from a fixed point by an elastic string, and is acted upon by no force except its weight and the supporting force exerted by the string. To determine the motion.

We must here make use of the property that *a stretched elastic string exerts a resisting force proportional directly to the amount of stretching.* This law applies not only to elastic strings but to a bar or rod of any elastic material. The law is established by experiment, and for many substances is nearly, though perhaps in no case exactly, true.

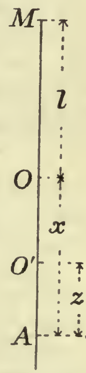


FIG. 106.

Let MO (Fig. 106) represent the unstretched string, M being the point of support. Let l denote the length MO, and x the amount of stretching at the time t; at which instant the end of the string, originally at O, is at A. The upward force exerted by the string upon the particle is then proportional to x, and we may put $P_2 = -cx$ in equation (1), c being the force necessary to produce an elongation of one unit of length. Since the force producing a given elongation is greater in proportion as the unstretched length is less, c is inversely proportional to l, and we may write

$$P_2 = -(e/l)x,$$

e being a constant.

For P_1 may be written mg, the weight of the body; hence equation (1) becomes

$$m(d^2x/dt^2) = mg - (e/l)x. \quad . \quad . \quad . \quad (2)$$

This equation can be integrated directly, but the process is simpler if the origin of coördinates is changed. Let O' (Fig. 106) be the position of the end of the string when the two forces just balance each other; the distance of this point from O is the value of x found by putting $P_1 + P_2 = 0$, that is

$$mg - ex/l = 0, \quad \text{or} \quad x = mgl/e.$$

Introducing a new variable z, such that

$$x = z + mgl/e,$$

equation (2) becomes

$$d^2z/dt^2 = -(e/ml)z.$$

The variable z evidently means the abscissa of the particle measured from O'. The last equation is identical in form with equation (1) of Art. 229, and the solution there given is here applicable. It is thus seen that in the case now under discussion the motion is the same as if the particle were acted upon by a single force directed toward the point O' and varying directly as the distance from that point.

Examples.

1. If a force equal to 10 pounds-weight would change the length of the string from 5 ft. (its natural length) to 6 ft., and if the mass of the body is 5 lbs., write the equation of motion.

Ans. $d^2x/dt^2 = (1 - 2x)g$, or $d^2z/dt^2 = -2gz$.

2. In the same case, let the body be at rest when the string is unstretched; determine the motion completely.

Ans. $z = -\frac{1}{2}\cos(t\sqrt{2g})$.

3. Can the initial conditions be such that the equation deduced above does not apply throughout the whole of the motion? If so, how could such a case be treated?

[The value of P_2, the upward pull of the string on the body, becomes zero when the string shortens to its natural length, the body being then at O (Fig. 106). If the body rises above this point, P_2 is zero, and so long as this is the case the body moves under the single force of gravity.

If the string be replaced by a rigid bar of elastic material (such as steel or wrought iron when stretched within the "limit of elasticity"), the force P_2 will follow the same law when the bar is shortened as when it is lengthened; that is, $-ex/l$ is the value of P_2 throughout the whole motion,* even if the initial conditions are such that the length of the bar becomes less than the natural length.]

* This assumes that the "modulus of elasticity" has the same value in compression as in tension.

4. With the data of Ex. 1, let the body have a velocity of 10 ft.-per-sec. at the instant when the string is unstretched. Determine the motion completely.

234. Motion in a Resisting Medium.— If a body be moving through the air, or through any fluid, the fluid exerts upon it forces which depend upon the velocity. The effect of these forces is to retard the motion of the body; in other words, the resistance of the medium is equivalent to a force whose direction is always opposite to that in which the body is moving. Hence in the general equation of rectilinear motion, such a resistance will enter as a force whose magnitude is some function of the velocity, and whose direction is opposite to that of the velocity; that is, for such a force,

$$P = f(v),$$

care being taken that the correct sign is used

EXAMPLES.

1. Write the equation of motion for a body moving vertically under gravity and the resistance of the air, the latter being assumed to vary directly as the velocity. Explain the meanings of any constants entering the equation. Integrate the equation completely.

2. In the same case, assume the resistance of the air to vary as the square of the velocity. Can a single equation be written for the whole motion in this case, if the body has initially an upward velocity? Integrate the equation once, thus determining the relation between velocity and time.

235. Motion of Connected Particles.— If two particles are connected by a string which is kept tight, each is acted upon by a force due to the string. If the string is without weight and is not in contact with any body except the two particles mentioned, the tension sustained by it has the same value at every cross-section. Hence the string exerts equal and opposite forces upon the particles.

If the string passes around smooth pegs or pulleys, the tension is still uniform throughout its length, and the forces exerted upon the particles by the string are equal though they may not be opposite.

If the equation of motion be written for each particle, each equation will contain the force due to the tension of the string. By combining the two equations this unknown force may be eliminated. This will be illustrated by a particular case.

Let two particles whose masses are m_1 and m_2 be connected by a

string which passes over a smooth pulley. Let the initial conditions be such that each particle describes a vertical line, one rising while the other falls. Assume the string to be without weight, perfectly flexible and inextensible. Assume also that the mass of the pulley is so small as to be negligible, and that it can revolve without frictional resistance. The arrangement is shown in Fig. 107.

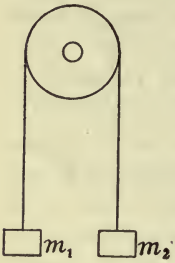


FIG. 107.

Let m_1 be greater than m_2, and assume the particles to be at rest at a certain instant. Then m_1 will fall and m_2 will rise. Take the position of rest as origin for each particle, and let x denote the distance of each from its origin at the time t. If $T =$ tension in string, the resultant force acting upon m_1 is $m_1g - T$ downward, and the resultant force acting upon m_2 is $T - m_2g$ upward. The acceleration of m_1 is $\ddot{x}$ downward, and that of m_2 is $\ddot{x}$ upward. The equations of motion for the two particles are therefore

$$m_1g - T = m_1\ddot{x}; \quad . \quad . \quad . \quad . \quad (1)$$

$$T - m_2g = m_2\ddot{x}. \quad . \quad . \quad . \quad . \quad (2)$$

The unknown quantity T may be eliminated by adding the two equations. The result is

$$(m_1 - m_2)g = (m_1 + m_2)\ddot{x}. \quad . \quad . \quad . \quad (3)$$

This equation is the same as would apply to the motion of a particle of mass $m_1 + m_2$ acted upon by a force $(m_1 - m_2)g$; *i. e.*, by a force equal to the difference between the weights of the particles. It is as if the combined mass of the two particles were being pulled in opposite directions by forces m_1g and m_2g.

Evidently equation (3) may be treated by the methods employed in Arts. 226 and 227. The equations derived from the motion of a falling body may be made applicable to the present case by substituting $[(m_1 - m_2)/(m_1 + m_2)]g$ for g.

Value of the tension.— By eliminating $\ddot{x}$ between any two of equations (1), (2) and (3), the value of T is found. It is

$$T = 2[m_1m_2/(m_1 + m_2)]g. \quad . \quad . \quad . \quad (4)$$

EXAMPLES.

1. If the two masses are 4 lbs. and 4.1 lbs., determine the acceleration, the tension, the velocity acquired after 0.5 sec., and the distance fallen through in 0.5 sec. *Ans.* Acceleration $= g/81$.

2. Two masses of 2 kilogr. each are suspended from the ends of a string which passes over a smooth pulley. The system being at rest, a mass of 10 gr. is added to one side. Determine the subsequent motion.

3. Three particles are connected by two strings, one of which passes over a smooth pulley as in Fig. 108. Determine the motion, and the tensions in the strings.

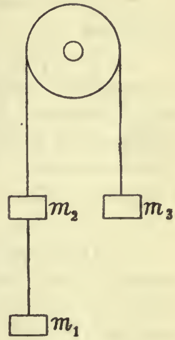


FIG. 108.

Let m_1, m_2 and m_3 denote the masses of the three particles, T_1 the tension in the string m_1m_2 (Fig. 108), and T_2 the tension in m_2m_3. Let x be the distance of m_3 below a certain fixed point; then x is also the distance of m_1 above some fixed point, and of m_2 above another fixed point. The equations of motion for the three particles are

$$T_1 - m_1g = m_1\ddot{x};$$
$$T_2 - T_1 - m_2g = m_2\ddot{x};$$
$$m_3g - T_2 = m_3\ddot{x}.$$

The unknown quantities T_1 and T_2 may be eliminated by adding the three equations. The result is

$$(m_3 - m_1 - m_2)g = (m_1 + m_2 + m_3)\ddot{x}.$$

This is identical with the equation of motion for a particle of mass $m_1 + m_2 + m_3$ acted upon by a force m_3g in one direction and by forces m_1g, m_2g in the opposite direction.

The values of T_1 and T_2 are

$$T_1 = m_1(g + \ddot{x}) = 2[m_1m_3/(m_1 + m_2 + m_3)]g;$$
$$T_2 = m_3(g - \ddot{x}) = 2[(m_1 + m_2)m_3/(m_1 + m_2 + m_3)]g.$$

4. In Ex. 3, let m_1, m_2 and m_3, expressed in kilograms, have values 2, 3 and 4. Determine the acceleration and the tensions.

Ans. $T_1 = 16/9$ kilograms-weight, $T_2 = 40/9$ kilograms-weight.

5. Let $m_1 = 4$ lbs., $m_2 = 3$ lbs., $m_3 = 2$ lbs. Determine T_1, T_2 and the acceleration.

Ans. The acceleration of m_1 is $5g/9$ downward.

MISCELLANEOUS EXAMPLES.

[In the following examples the student should in every case write the differential equation of motion, even if it is not found possible to complete the solution. He should also examine what conditions are needed for the determination of the constants of integration. If this is done, the problem in Dynamics is reduced to one in mathematical analysis.]

1. Two particles whose masses are equal are connected by an elastic string. They are projected from the same point with equal and opposite velocities. Determine the motion, assuming no forces to act except those due to the tension of the string.

Ans. Let a be the natural length of the string and k the force necessary to double its length. If A is the position of one particle when the string begins to stretch, that particle describes half of a harmonic oscillation (Art. 229) about A, returning to A after a time $\pi\sqrt{(am/2k)}$. The two particles then approach each other with constant velocity.

2. Assume the conditions as in Ex. 1, except that the masses are unequal.

3. Two particles whose masses are m_1 and m_2, connected by an inextensible string, are suspended from a fixed point by an elastic string attached to m_1. If the system is held at rest with the elastic string at its natural length and then released, determine the subsequent motion. Determine also the tension in the lower string in the lowest position.

Ans. The required tension is twice the weight of m_2.

4. Two particles, each of mass m, are connected by an elastic string whose natural length is l. The force necessary to double the length of the string is F. The particles repel each other with forces varying directly as their distance apart; the repulsive force being F' when the distance is l. If both particles are held at rest at a distance apart equal to l and are then released, determine the subsequent motion.

Ans. If $F > F'$, the motion of either particle is given by the equation

$$\frac{x}{l} = \frac{F + F' \sin\left[t\sqrt{2(F - F')/ml}\right]}{2(F - F')}.$$

The motion is a harmonic oscillation (Art. 229) about a point distant $Fl/2(F - F')$ from the point midway between the particles, the time of a complete oscillation being $\pi\sqrt{[2ml/(F - F')]}$. If the origin be taken at the center about which the oscillation takes place, the motion is given by equation (3) of Art. 229, if $k = 2(F - F')/ml$. If $F' > F$, the motion reduces to a case under Art. 230.

5. Solve with data as in Ex. 4 except that the masses are unequal.

6. In Ex. 4 let the mass of each particle be 40 gr.; let $l = 20$ c.m., $F = 50$ grams-weight, $F' = 30$ grams-weight. Determine the period of a complete oscillation, and the range of motion.

Ans. Time of oscillation $= 0.9$ sec.

7. In Fig. 108 let the string m_1m_2 be elastic, its unstretched length being l. Let the system be initially at rest with $m_1m_2 = l$. Determine the subsequent motion.

8. Two particles, each of mass m, repel each other with forces varying inversely as the square of their distance apart. When the distance is l the force is F. They are held at rest in a vertical line

at a distance apart l', and are then released. Determine their subsequent motion, assuming the only forces acting on the particles to be gravity and their mutual repulsion.

Ans. If r is the distance of the particles apart at any time, their velocity of separation dr/dt is given by the equation

$$(dr/dt)^2 = (4Fl^2/m)(1/l' - 1/r).$$

This equation can be integrated, giving t as a function of r. The center of gravity of the two particles falls a distance $\frac{1}{2}gt^2$ in t sec. These two relations determine the motion.

9. A particle of mass m is attached to one end of an elastic string whose natural length is l. The other end of the string is attached to a fixed support. The particle is dropped from this fixed point of attachment. Determine the motion. Let the force necessary to stretch the string to double its natural length be F.

10. The natural length of an elastic string is l; under a pull F it stretches to a length $2l$. The ends are attached to fixed pegs whose distance apart is $3l/2$. A particle of mass m is attached to the string at a point distant $4l/5$ from one peg, and is forcibly brought to the point midway between the pegs and then released. Determine the time of oscillation. *Ans.* $(4\pi/15)\sqrt{(14ml/F)}$.

11. A body is projected into a resisting medium which exerts a retarding force proportional to the velocity. If no other force acts upon the body, determine the motion. Let m = mass, V = initial velocity, and let the force = F when the velocity = V.

Ans. Let x' denote the distance the body would move, and t' the time it would move, before coming to rest against a constant force F. Then the distance described in time t against the actual force is $x = 2x'(1 - e^{-t/t'})$. It is seen that x approaches the limit $2x'$, but the particle never comes to rest.

12. A mass of 5 kilogr. moves in such a way that its acceleration is directly proportional to its velocity but has the opposite direction. When expressed in centimeter-second units the velocity and acceleration are numerically equal. In a certain position the velocity is 50 c.m.-per-sec. Determine the value of the force after the particle has moved 40 c.m. from this position. *Ans.* 50,000 dynes.

13. In Ex. 12, determine the value of the force 1 sec. after the instant at which the velocity is 50 c.m.-per-sec. When will the particle come to rest, and how far will it move?

Ans. 92,000 dynes. The distance passed over will approach 50 c.m. as a limit, but the particle will never come to rest.

CHAPTER XIV.

MOTION IN A CURVED PATH.

§ 1. *Position, Displacement and Velocity.*

236. Direction.— When the path of a particle is a straight line, its motion has at every instant one of two opposite directions. These two directions may be fully specified by signs plus and minus. But when the path is a curve, the direction of the motion continually changes, and cannot be specified so simply.

In the foregoing analysis of the motion of a particle whose path is a straight line, definitions have been given of displacement, velocity and acceleration. These definitions must now be enlarged. Each one of these quantities is at every instant associated with a definite *direction* in space, and this *direction* is an essential element in its value. The quantities named are, in fact, *vector quantities* (Art. 16).

237. Position Given by Vector.— If a particle is not confined to a straight line, but has any motion in space, its position at any instant may be specified by means of a *vector of position.*

If P (Fig. 109) is the position of the particle at any instant, and O any fixed point taken as origin of reference, the position of P is known if the vector OP is known. For, to know the vector OP completely is to know (1) the direction from O to P and (2) the length of the line OP. Knowing these, P may be located from O. OP is called the *position-vector* of the particle.

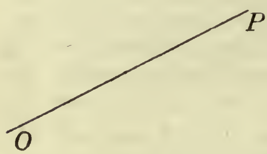


FIG. 109.

As the particle moves, the position-vector varies either in length, or in direction, or in both length and direction.

Practically, in order to describe completely the magnitude and direction of a vector, the values of certain angles and distances, measured from definite lines, planes or points regarded as fixed, must be given. For the present purpose it is not necessary to consider how the value of a vector may best be specified in practice.

238. Displacement.— If the particle moves from A to B during any interval of time, the vector AB is its *displacement.*

This definition is independent of the actual path followed, which may be any line joining A and B. If the path is the straight line AB (Fig. 110), the vector AB is the *actual* displacement. If any other path ACB is followed, the vector AB is still regarded as the *total* or *resultant* displacement.

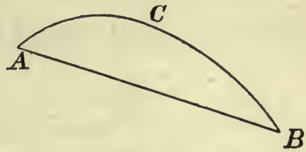


FIG. 110.

239. **Uniform Motion.**— The motion is said to be *uniform* when the particle receives equal displacements in any equal intervals of time, however these intervals be chosen.

It is to be particularly noticed that, since displacement is a vector quantity, successive displacements are not equal unless they agree in *direction* as well as in magnitude. Hence, uniform motion as here defined has a constant direction; *i. e.*, it is rectilinear. This case of motion has been considered in Chapters XII and XIII.

240. **Variable Motion.**— If the particle receives unequal displacements in equal intervals of time, the motion is *variable.* The unequal displacements may differ in length only, in direction only, or in both length and direction.

If the successive displacements agree in direction, the path is a straight line, and the motion falls under the case already treated. The general analysis which follows includes this as a special case.

241. **Velocity.**— In case of motion not confined to a straight line, the velocity of a particle may still be defined as its *rate of displacement;** but the definition must be interpreted with reference to the meaning of displacement as a vector quantity.

Whatever path a particle may describe, it is at every instant moving at a definite rate and in a definite direction. The velocity is a vector quantity whose direction coincides with that in which the particle is moving, and whose magnitude measures the rate of motion.

The meaning of the definition of velocity and the method of estimating its value are best understood by considering "average" velocity in case of the unrestricted motion of a particle.

242. **Average Velocity.**— The *average velocity* of a particle for an interval of time during which it receives any displacement, is the

*As in rectilinear motion, Art. 190.

velocity of a particle which, moving uniformly, would receive the same total displacement in the same time.

The average velocity is a vector quantity, its direction being that of the total displacement for the interval.

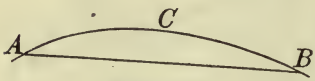


FIG. 111.

Note that this definition is independent of the path. Thus, if the particle moves from A to B (Fig. 111) along any path ACB, the *total* displacement is the vector AB. If t_1 and t_2 are the values of t corresponding to the positions A and B, the average velocity is given by the expression

$$(\text{vector } AB)/(t_2 - t_1),$$

whatever the form of the path ACB.

243. Approximate Value of Velocity at an Instant.— An approximate value of the velocity of a particle at any instant may be found by determining the average velocity for a very short time. Thus, let A (Fig. 111) be the position of the particle at the time t, and B its position after a short interval Δt; then

$$(\text{vector } AB)/\Delta t$$

is the average velocity for the interval Δt, and is an approximate value of the velocity in the position A. The approximation is closer the shorter the interval Δt.

244. Exact Value of Velocity at an Instant.— The true value of the velocity at the instant t, when the particle is at A, is the limit approached by the above approximate value as Δt approaches zero. That is

$$\text{velocity at } A = \text{limit } [(\text{vector } AB)/\Delta t].$$

This limit is a vector quantity.

(1) Its direction is that of the tangent to the path at A, since the chord AB approaches the tangent as B approaches A.

(2) Its magnitude is ds/dt, if s denotes the length of the path measured from some fixed point to the position of the particle. For as the point B approaches A, the chord AB and the arc AB approach a ratio of equality, so that

$$\text{limit } [(\text{chord } AB)/\Delta t] = \text{limit } [(\text{arc } AB)/\Delta t] = ds/dt.$$

The magnitude of the velocity, considered without reference to

direction, is called the *speed*. If this is denoted by v, its value is always given by the equation

$$v = ds/dt.$$

245. Graphical Representation of Velocity; Hodograph.— Let AB (Fig. 112) be the path of a particle, described in any manner. From some point O' draw $O'A'$ to represent in magnitude and direction the velocity of the particle in the position A, and $O'B'$ to represent the velocity in the position B. Also, for every intermediate position of the particle, as P, draw a vector $O'P'$ representing the velocity of the particle when in that position. Through the extremities of all such vectors draw a curve. This is called the *curve of velocities* or *hodograph* of the motion.

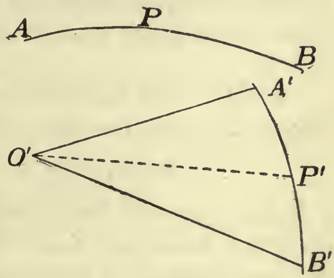


FIG. 112.

EXAMPLES.

1. A particle describes a circle of 2 ft. radius with uniform speed, the whole circumference being described in half a second. Required (a) the speed; (b) the values of the average velocity for 1/10 sec. and for 1/8 sec.; (c) the values (both direction and magnitude) of the instantaneous velocity at two instants 1/10 sec. apart.

2. A particle describes a circle of radius r with uniform speed v. Determine the magnitude and direction of the average velocity for an interval t. *Ans.* Its magnitude is $(2r/t) \sin (vt/2r)$.

3. A particle describes a circle of radius r in such a way that $s = at^2$, s being the length of the path described in time t, and a being a constant. Required (a) the speed at any time; (b) the average velocity during the interval from t_1 to t_2.

4. In Ex. 3, let $r = 60$ c.m., and let the length of arc described during the first second be 120 c.m. Required (a) the speed at the end of 2 sec.; (b) the average velocity during an interval of 0.1 sec. after $t = 2$. *Ans.* (a) 480 c.m.-per-sec. (b) 478.5 c.m.-per-sec.

5. Draw the hodograph for the motion described in Ex. 1.

6. Draw the hodograph for the motion described in Ex. 4.

7. A particle describes any curved path with uniform speed v. What is the form of the hodograph?

§ 2. *Velocity-Increment and Acceleration.*

246. Increment of Velocity.—Let the values of the velocity of a moving particle at two instants t_1 and t_2 be represented by the vectors $O'A'$ and $O'B'$ respectively (Fig. 113). Then the vector $A'B'$ is the *velocity-increment* for the interval from t_1 to t_2. For $A'B'$ is the vector which must be added to $O'A'$ to produce $O'B'$. (See Art. 22.)

It will be seen that velocity-increment as thus defined is not the amount by which the *speed* changes. The increment of the *speed* is

$$\text{length } O'B' - \text{length } O'A' = v_2 - v_1,$$

if v_1 and v_2 are the initial and final values of the speed. It is only when the initial and final velocities are parallel that the *velocity-increment* is equal in magnitude to $v_2 - v_1$.

247. Acceleration.—If a particle is moving in a curved path, the acceleration may be defined as the *rate of change of the velocity*, just as in the case of rectilinear motion (Art. 203). But since "change of velocity" (or velocity-increment) is a vector quantity, acceleration is also a vector quantity; and in computing its value both direction and magnitude must be considered.

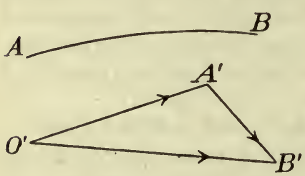


FIG. 113.

In this definition of acceleration, "rate of change of velocity" must be understood to mean "increment of velocity per unit time."

Consider first the case in which the velocity is varying uniformly; by this is meant that the increments of velocity during any different intervals of time have the same direction and are proportional to the intervals. Thus, let the vector $A'B'$ (Fig. 113) represent the velocity-increment during an interval of time Δt, and let the velocity-increments during any partial intervals into which Δt may be divided have the direction $A'B'$ and be proportional in magnitude to the partial intervals. Then the acceleration as above defined has a constant value throughout the time Δt, that value being

$$(\text{vector } A'B')/\Delta t.$$

When the velocity is not varying in this uniform manner, the

acceleration is a variable vector quantity, having a definite magnitude and direction *at any instant.* The exact meaning of the definition of acceleration in this case, and the method of computing its value, may be best understood by a consideration of "average acceleration."

248. Average Acceleration.—The *average acceleration* of a particle for an interval of time during which the velocity varies in any manner is an acceleration which, if constant in magnitude and direction, would result in the same velocity-increment in the same time.* The value of the average acceleration is found by dividing the velocity-increment received during the whole interval by the duration of the interval.

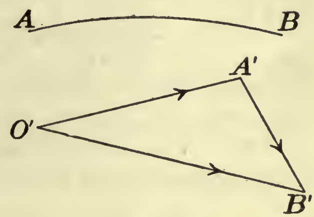


FIG. 114.

If $A'B'$ (Fig. 114) represents the velocity-increment for the interval from t_1 to t_2, the expression

$$(\text{vector } A'B')/(t_2 - t_1) \quad \text{or} \quad (\text{vector } A'B')/\Delta t$$

gives the average acceleration for that interval.

This expression cannot be reduced, as in the case of rectilinear motion (Art. 208), to the form

$$(v_2 - v_1)/(t_2 - t_1),$$

in which v_1 and v_2 denote the initial and final values of the magnitude of the velocity; for it is only when the two velocities are parallel that $v_2 - v_1$ gives the velocity-increment.†

249. Approximate Value of Acceleration at an Instant.—The average acceleration for a short interval immediately following any instant is an approximate value of the true acceleration *at* that instant. Thus, if the velocity at the given instant is represented by $O'A'$

* Compare Art. 208.

† If, however, v_1 and v_2 be regarded as vector symbols representing the initial and final values of the velocity in direction as well as in magnitude, the vector $v_2 - v_1$ is the value of the velocity-increment (Art. 22) and

$$[\text{vector } (v_2 - v_1)]/(t_2 - t_1)$$

is the value of the average acceleration.

(Fig. 114), and the velocity after the short interval Δt by $O'B'$, then

$$(\text{vector } A'B')/\Delta t$$

is an approximate value of the required acceleration. The approximation is closer the shorter the interval Δt.

250. Exact Value of Acceleration at an Instant.— The true value of the acceleration at the instant t is the limit approached by the above approximate value as the interval Δt approaches zero. That is,

$$\text{acceleration} = \text{limit } [(\text{vector } A'B')/\Delta t].$$

To determine this limiting value, the following reasoning may be employed:

Let $O'A'$ and $O'B'$ (Fig. 115) represent the values of the velocity at the beginning and end of Δt respectively, so that vector $A'B'$ denotes the increment of velocity, the curve $A'B'$ being the hodograph or curve of velocities (Art. 245). As Δt approaches 0, the point B' approaches A' along the curve $A'B'$. The limiting *direction* of

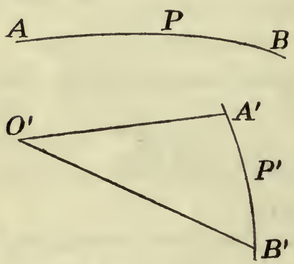


FIG. 115.

$$(\text{vector } A'B')/\Delta t$$

is that of the tangent to the curve at A'. Also, the limit of the magnitude of this vector quantity is the derivative of s' with respect to t, if s' is the length of the curve $A'B'$ measured from some fixed point up to the point which is describing it; that is

$$\text{limit } [(\text{length } A'B')/\Delta t] = ds'/dt.$$

Consider the curve $A'B'$ (Fig. 115) to be described by a moving particle whose position at every instant corresponds to that of the given particle describing the path AB; so that if P represents the position of the latter at any instant, and the vector $O'P'$ its velocity, P' is the position of the point describing the hodograph or curve $A'B'$. Evidently, ds'/dt is the velocity of the point P'. Hence

The acceleration of a point describing any path in any manner is at every instant equal (in magnitude and direction) to the velocity of the point describing the hodograph.

The direction of the acceleration is always inclined toward the concave side of the path, but nothing further can in general be said

as to its direction relative to the path. Neither can any simple expression be given for its magnitude in terms of the coördinates of the path.*

251. Sudden Change of Velocity.— In what precedes it has been assumed that when the velocity changes from one value to another, it passes through a continuous series of intermediate values, and that for a finite change of velocity a finite interval of time is required. It is conceivable that this condition might be violated, and that a particle might receive an *instantaneous* increment of velocity. In case of the motions actually occurring in nature there is no reason to believe that instantaneous changes of velocity occur. *Sudden* changes of velocity do, however, occur and will be treated in another place. (See Chapter XVI.)

Examples.

1. A particle describes a circle of radius 2 ft. with a uniform speed such that the entire circumference is described in 1 sec. Required (*a*) the velocity-increment for an interval of 0.1 sec.; (*b*) the average acceleration for the same interval (give its magnitude and the angle its direction makes with that of the initial velocity).

Ans. (*a*) 7.766 ft.-per-sec.; (*b*) 77.66 ft.-per-sec.-per-sec.; angle $= 108°$.

2. A particle describes a circle of radius r with uniform speed v. Required (*a*) the magnitude and direction of the increment received by the velocity while the particle describes one-fourth of the circumference; (*b*) the magnitude and direction of the average acceleration for the same interval.

Ans. (*b*) $(2\sqrt{2}/\pi)(v^2/r)$, inclined $135°$ to initial velocity.

3. With data as in Ex. 2, determine the average acceleration in any finite interval Δt. (Give its magnitude and the direction it makes with the initial velocity.)

Let AB (Fig. 116) represent the arc described in time Δt. The velocity at the beginning of the interval is represented by a vector $O'A'$, of length v, perpendicular to the radius OA; the velocity at the end of the interval is represented by a vector $O'B'$, also of length

---

*If v is regarded as a vector symbol representing the velocity in direction as well as in magnitude, and if Δv is the increment of the vector v in the time Δt, the value of the acceleration may be written

$$p = \text{limit } [(\text{vector } \Delta v/\Delta t)] = \text{vector } (dv/dt).$$

But the meaning of dv/dt is quite different from that of the same expression when v denotes the *speed*.

v, perpendicular to the radius OB. The velocity-increment is therefore vector $A'B'$, and the average acceleration is

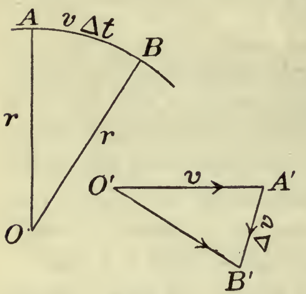


FIG. 116.

$$(\text{vector } A'B')/\Delta t.$$

The length $A'B'$ is $2v \sin (A'O'B'/2)$. But (expressing angles in radians),

$$\text{angle } A'O'B' = \text{angle } AOB$$
$$= (\text{arc } AB)/OA = (v\Delta t)/r;$$

hence

$$\text{length } A'B' = 2v \sin (v\Delta t/2r),$$

and the average acceleration has the magnitude

$$A'B'/\Delta t = (2v/\Delta t) \sin (v\Delta t/2r).$$

Its direction is perpendicular to the chord AB.

4. Taking the result reached in Ex. 3, let the interval Δt approach the limit zero; determine the magnitude and direction of the limiting value of the average acceleration. Determine thus the exact value of the acceleration at any instant.

As Δt approaches zero, the chord AB approaches the tangent to the circle at A, and the direction of the vector $A'B'$, which is perpendicular to the chord AB, approaches AO. Also, since the angle AOB ($= A'O'B'$) approaches zero as Δt approaches zero, the value of the angle in radians may in the limit replace the sine; that is, $v\Delta t/2r$ may replace $\sin (v\Delta t/2r)$. Therefore

$$\text{limit } [A'B'/\Delta t] = \text{limit } [(2v/\Delta t)(v\Delta t/2r)] = v^2/r.$$

That is, the actual acceleration at the instant when the particle is at A is directed toward the center of the circle and has the magnitude v^2/r.

5. What is the value of the acceleration of a particle moving as described in Ex. 1?

Ans. 78.9 ft.-per-sec.-per-sec. toward the center.

6. Draw the path and the hodograph for the motion described in Ex. 2. Choosing any point P on the path, find the corresponding point P' on the hodograph. If P moves with the moving particle, what is the velocity of P' at any instant? From the principle stated in Art. 250, that the velocity of P' is equal in magnitude and direction to the acceleration of P, verify the result of Ex. 4.

7. A particle describes the perimeter of a regular hexagon with uniform speed v. It receives what sudden increment of velocity as it passes an angle of the polygon?

8. What sudden increments of velocity are received by a particle describing the perimeter of a regular polygon of n sides with uniform speed v?

§ 3. *Motion and Force.*

252. Effect of a Constant Force.—The effect of a constant force upon the motion of a body which initially is either at rest or moving along the line of action of the force has been considered in Arts. 214 and 215. The principle there stated is that the force produces a velocity-increment proportional directly to the force and to the time during which it acts, and inversely to the mass of the body. The case in which the body has initially a velocity not parallel to the force must now be considered.

If a moving particle be acted upon by a force whose direction is inclined to that of the motion, there results a continuous change in the direction of the motion. Thus, suppose a particle to move along the straight line AB (Fig. 117) with uniform velocity, not being acted upon by any force until arriving at the point B. Let it then be acted upon by a force represented by the vector MN, which remains constant in magnitude and direction for a period Δt. The particle is deflected from the original straight path AB, and describes some curve BC. The velocity thus changes in direction, and in general it changes also in magnitude. Representing the velocity at the beginning of the interval Δt by a vector $O'B'$ (Fig. 117) parallel to AB, and the velocity at the end of the interval by a vector $O'C'$ parallel to the tangent to the path BC at C, the total change of velocity (or velocity-increment) is represented by the vector $B'C'$. The magnitude and direction of this velocity-increment depend upon (1) the magnitude and direction of the force, (2) the duration of the interval Δt, and (3) the mass of the particle.

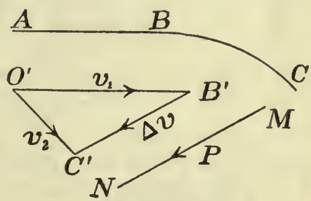


FIG. 117.

(1) The direction of the velocity-increment agrees with that of the force, and its magnitude is proportional directly to the magnitude of the force.

(2) Its magnitude is proportional directly to the duration of the interval.

(3) Its magnitude is proportional inversely to the mass of the particle.

These principles may be stated in the following general proposition:

A constant force acting upon a particle gives it a velocity-increment whose direction is that of the force, and whose magnitude is proportional directly to the force and to the time during which it acts and inversely to the mass of the particle.

This proposition is identical in form with that given in Art. 215. But since the discussion was there restricted to the case of rectilinear motion, the velocity-increment was necessarily parallel to the line of motion. This restriction being removed, the proposition is still true, *velocity and velocity-increment being regarded as vector quantities.* The special case of rectilinear motion is included in the general statement.

It is thus seen that the effect of a constant force upon the motion of a particle is estimated in the same way, whatever the initial velocity of the particle. The velocity-increment is in every case equal

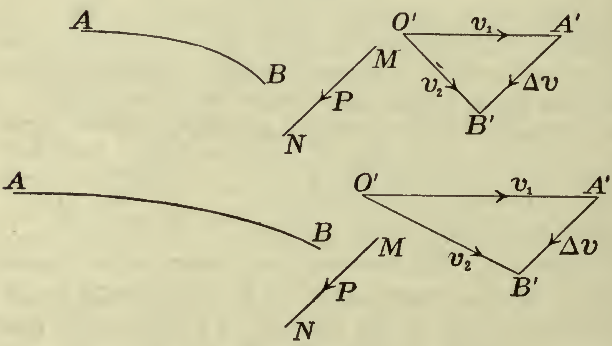


FIG. 118.

to the velocity which would be produced by the same force acting for the same time on the same particle initially at rest. The *final velocity* is the vector sum of this velocity-increment and the initial velocity.

The path of the particle will depend both upon the initial velocity and upon the magnitude and direction of the force. To illustrate this, compare two cases in which the force has the same value while the initial velocity has different values. The two cases are shown in Fig. 118. In each case the initial velocity v_1 is represented by the vector $O'A'$, the direction being the same in both cases but the

magnitude being much greater in the second case than in the first. The force (represented by the vector MN) has the same magnitude and direction in both cases. The mass of the particle is also the same, so that the values of the velocity-increment in a given time are equal in the two cases. The velocity-increment being represented by the vector $A'B'$, the final velocity v_2 is given by the vector $O'B'$. The values of v_2 in the two cases differ both in magnitude and in direction. The two paths are also quite different. If A and B are the initial and final positions of the particle, the tangent at A is parallel to v_1 and the tangent at B is parallel to v_2. The curvature of the path is evidently greater in the first case than in the second; while the length of the path is greater in the second case, because the speed is greater throughout the interval.

Examples.

1. A body is projected horizontally with a velocity of 50 ft.-per-sec., after which it is acted upon by the constant force of gravity. Determine the magnitude and direction of its velocity at the end of 0.5 sec., 1 sec. and 2 sec.

[The velocity-increment due to gravity is to be computed as if the body fell vertically from rest. See Art. 227.]

2. A body is projected with a velocity of 50 ft.-per-sec. in a direction inclined 40° upward from the horizontal. Determine the magnitude and direction of the velocity at the end of 0.5 sec., 1 sec. and 2 sec.

Ans. At the end of 2 sec. the velocity is 50 ft.-per-sec., directed 40° downward from the horizontal (very nearly).

3. A body falling vertically at the rate of 30 ft.-per-sec. receives a blow in a horizontal direction which, if the body had been at rest, would have given it a velocity of 100 ft.-per-sec. Determine the magnitude and direction of the velocity immediately after the blow.

4. With data of Ex. 3, determine the magnitude and direction of the velocity 1 sec. after the blow, assuming gravity to be the only force acting on the body.

Ans. 117.8 ft.-per-sec. inclined 31° 56′ downward from the horizontal.

253. Equation of Motion for Particle Acted Upon by a Constant Force.—The general principle above stated (Art. 252) may be expressed by an equation similar to that given in Art. 216 for the case of rectilinear motion. Let the increment of velocity (vector $A'B'$, Fig. 118) be represented by Δv, and the force (vector MN, Fig. 118) by P. Then if m is the mass of the particle and Δt the

time in which the constant force P produces the velocity-increment Δv, the general principle is equivalent to the equation

$$\Delta v = k(P\,\Delta t/m).$$

But this must be interpreted as a *vector* equation, expressing identity of direction as well as equality of magnitude.

Taking units of force, mass, length and time as in Art. 217 or Art. 218, so that $k = 1$, the equation may be written

$$P = m(\Delta v/\Delta t).$$

But $\Delta v/\Delta t$, the increment of velocity per unit time, is the acceleration (Art. 247). Hence, if acceleration is designated by p, the equation may be written

$$P = mp, \quad \text{or} \quad p = P/m.$$

Unless otherwise stated, it will hereafter be assumed that units are so chosen that $k = 1$.

254. Effect of a Variable Force.—A particle may be acted upon by a force which varies in magnitude, in direction, or in both magnitude and direction. Consider the effect of such a variable force upon the motion of the particle.

Let the vector $O'A'$ (Fig. 114) represent the velocity of the particle at an instant t, and the vector $O'B'$ the velocity after an interval Δt. Then $A'B'$ is the velocity-increment received by the particle during the interval.

The value of a constant force which would produce the same velocity-increment in the same time is (by Art. 253)

$$m\,(\text{vector } A'B')/\Delta t.$$

If Δt is very small, the actual force differs little from this value during the interval; and the true value of the force at the beginning of the interval is the limit approached by this approximate value as Δt approaches zero. That is, if P is the value of the force and p that of the acceleration at the beginning of the interval,

$$P = \text{limit } [m\,(\text{vector } A'B')/\Delta t] = mp.$$

Here P and p must be understood as vector symbols, and the equation expresses equality both in magnitude and in direction.

Thus the equation of motion has the same form when the force is variable as when it is constant.

255. Resultant of Two or More Forces Acting Simultaneously Upon the Same Particle.—Any number of forces acting simultaneously upon a particle are equivalent, in their effect upon the motion, to a single force. For the acceleration has at every instant a definite value p, and this same acceleration would result from the action of a single force of magnitude mp, agreeing in direction with p.

A single force which, acting alone, has the same effect as several forces acting simultaneously, is called their *resultant* (Art. 55).

The resultant of any number of forces acting at the same time upon a particle is a force equal to their vector sum.

This proposition, for the case of two forces, expresses the principle of the parallelogram of forces. Assuming its truth in case of two forces, its truth for any number of forces follows immediately. Its meaning may be explained as follows:

Let p_1 be the acceleration due to a force P_1 acting alone upon a given particle, and p_2 the acceleration due to a force P_2 acting alone upon the same particle. Then if the particle be acted upon by P_1 and P_2 at the same time, its acceleration is the vector sum of p_1 and p_2. If the particle be acted upon at the same time by a third force P_3 which, acting alone, would produce acceleration p_3, the actual acceleration is the vector sum of p_1, p_2 and p_3. And similarly for any number of forces.

Thus, the acceleration due to the concurrent action of several forces may be computed in either of two ways,—(1) by determining the acceleration due to each acting alone and taking the vector sum of these separate accelerations, or (2) by determining the acceleration due to a single force equal to the vector sum of the several forces.

This principle must be regarded as a fundamental law of Dynamics, derived from experience and not deducible from any simpler law or laws.

256. Equation of Motion for Particle Acted Upon by Any Number of Forces.—The equation of motion may be written in the same form when several forces act upon the particle as when only a single force acts. It is

$$P = mp,$$

if m is the mass of the particle, p its acceleration at any instant, and P the vector sum of all forces acting upon the particle at that instant.

257. Remarks on General Equation of Motion.—The equation

$$P = mp \quad . \quad . \quad . \quad . \quad . \quad (1)$$

may thus be called the *general equation of motion* for a particle acted upon by any forces. In interpreting it, P may be taken to mean any single force and p the acceleration due to that force; or P may mean the resultant of several forces and p the acceleration due to their combined action. In either case it is to be understood that P and p are vector quantities having the same direction.

It must also be remembered that unless the units of force, mass, length and time are related in a certain manner (Art. 217), the equation will be

$$p = k(P/m), \quad \text{or} \quad P = k'mp.$$

The constant k (or k') will always be unity provided the unit force is taken as *that force which will give to the unit mass the unit acceleration.*

Thus, equation (1) may be used if the units are the foot, pound-mass, second and poundal; or the centimeter, gram, second and dyne. (Art 217.)

Or, any three of the units involved may be chosen arbitrarily, and the other determined in such a way as to make the constant k unity. In particular, the "engineers' kinetic system" of units, described in Art. 218, may be employed. That is, in using the equation $P = mp$, we may take the pound-force as the unit force, provided the mass be expressed in units, one of which is equal to g pounds-mass. If the mass of the body is known in pounds, its value in these new units is found by dividing by g. Thus, if M denotes the mass in pounds, the equation of motion will be $P = (M/g)p$.

258. Absolute Unit of Force.—The unit force which was usually employed in the earlier chapters of this book was the weight of a unit mass. The most precise method of determining the magnitude of a force practically consists in comparing it with this unit. Since, however, the weight of a given mass changes with its position on the earth's surface, forces determined experimentally in different localities in terms of this unit cannot be accurately compared unless the ratio between the weights of the same mass in the two localities is known.

In Art. 217 another unit of force was defined,—the force which, acting upon the unit mass, produces the unit acceleration. This has been called an *absolute* unit, since it is not dependent upon any particular force such as the weight of a body, nor upon the place at which its value is determined.

A force whose value is known in terms of the weight of a unit

mass at a given place may be expressed in terms of the absolute unit if the value of g (the acceleration due to gravity) at that place is known. Thus, a force equal to the weight of m pounds-mass at a certain place is equal to mg poundals, if g is the acceleration due to gravity at that place, expressed in feet-per-second-per-second. (Art. 227.) The most accurate method of determining the value of g at any place is to determine the time of vibration of a pendulum of known length. (Arts. 309 and 425.)

This definition of the absolute unit of force depends upon the effect of force in accelerating mass. Any accurate definition of force as a measurable quantity must rest upon the same basis. For this reason the general proposition or "law of motion" stated in Art. 252 is often regarded as a definition of force rather than as a proposition about forces. (See Art. 260.)

259. **Newton's Laws of Motion.**—The fundamental laws of motion, as formulated by Newton,* are as follows:

Law I. Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.

Law II. Change of motion is proportional to impressed force, and takes place in the direction of the straight line in which the force acts.

Law III. To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal and oppositely directed.

These laws include all the principles upon which the foregoing discussions have been based.

Law I expresses the principle stated in Arts. 31 and 213. This is often called the *law of inertia;* inertia being defined as that property of matter by virtue of which a body cannot of itself change its own state of motion.

Law II includes in its meaning the principles stated in Arts. 252–256. In order that this may be clearly understood, some explanation of the terms employed by Newton is needed.

The word *motion* is to be understood as meaning what is defined as *momentum* (Art. 312); that is, a quantity proportional directly to the mass of a body and to its velocity at any instant. *Change of*

* These laws are stated nearly in the language of Kelvin and Tait's "Natural Philosophy," §§ 244, 251, 261.

motion is therefore *change of momentum*, and is a vector quantity proportional to the mass and to the change of velocity or velocity-increment. Again, *impressed force* must be understood to mean, not simply force as the term is now understood and has been used in the foregoing discussions, but a quantity proportional to the force and to the time during which it acts. Thus, if a force of magnitude P acts for an interval of time Δt upon a body of mass m, and if Δv represents the resulting velocity-increment, the law asserts that $m\Delta v$ is proportional to $P\Delta t$. This is evidently the same result as that stated in Art. 252 for the case of a constant force, and extended to the case of a variable force in Art. 254.

That Law II includes also the law of composition of forces acting simultaneously (Art. 255) is not immediately evident from the language in which it is expressed. By Newton, however, the law was doubtless intended to imply that every force acting upon a body produces its effect independently of the action of other forces upon the same body, and that these effects (accelerations) combine according to the law of vector addition.*

There is some ground for the opinion that the law of composition of forces constitutes an independent principle among the fundamental laws of Dynamics. Whether this be the correct view or not, it is desirable that this law should receive explicit statement. Such a statement is given in Art. 255.

Law III has been stated and explained in Art. 35. In spite of its importance and its simplicity, its true meaning has often been missed.

In the explanation of this law given in Art. 35, the word *action* was interpreted as meaning *force*, thus limiting the law to a statement regarding the forces exerted by two bodies upon each other. By Newton the law was applied in a wider sense, as indicated by the scholium † which he added to it. This scholium is here omitted, since its full meaning cannot be explained without anticipating the results of a later discussion. The law will for the present be understood as referring only to forces. So long as the attention is confined to the motion of a single particle, no use is made of the third law. But it is of fundamental importance in the analysis of the motions of two or more particles which exert forces upon one another. The importance of the law in Statics has already been seen in Chapter VI.

* See Kelvin and Tait's "Natural Philosophy," §§ 254, 257.

† Thomson and Tait's "Natural Philosophy," § 263.

Examples.

1. A constant force acting upon a body of mass 40 lbs. for 3 sec. changes its velocity from 20 ft.-per-sec. north to 80 ft.-per-sec. east. Required the magnitude of the force in poundals.

Ans. 1099 poundals.

2. If the same force continues to act, what will be the velocity of the body after 3 more sec.?

Ans. 161.3 ft.-per-sec. S. 82° 52′ E.

3. A particle of 5 lbs. mass is acted upon by two constant forces at right angles to each other; one equal to the weight of 1 lb., the other equal to 20 poundals. What single force would produce the same effect? *Ans.* Magnitude of force = 37.9 poundals.

4. In Ex. 3, if the particle has initially a velocity of 20 ft.-per-sec. in a direction opposite to that of the lesser force, determine its velocity after 2 sec.

Ans. 17.6 ft.-per-sec. at angle of 47° 2′ with initial velocity.

5. A particle whose mass is 250 gr. is acted upon by two constant forces whose directions are inclined to each other at an angle of 60°. One force is equal to the weight of the particle, the other is equal to 50,000 dynes. If the velocity at a certain instant is 600 c.m.-per-sec. in a direction bisecting the angle between the forces, determine the velocity 3 sec. later.

6. In Ex. 5, what single force may replace the given forces?

7. If a body of 38 lbs. mass describes a circle of 40 ft. diameter at the uniform rate of 30 ft.-per-sec., what is the magnitude and direction of the resultant force acting upon it at any instant?

[Use result of Ex. 4, following Art. 251.]

8. A body of mass m describes a circle of radius r at the uniform rate of v ft.-per-sec. Required the magnitude and direction of the resultant force acting upon it at any instant.

260. Laws of Motion Regarded as Definitions of Force and Mass.— In the foregoing exposition of the laws of motion, it has been tacitly assumed that force and mass are quantities each of which is measurable independently of the other; so that either or both may be made fundamental in the system of units which we may choose to adopt. (Art. 216.) But when it is attempted to define these quantities with scientific accuracy, it appears to be impossible to avoid making use of the laws of motion themselves. It has already been shown that the second law furnishes a means of defining the unit force in an exact manner when the unit mass has been chosen (Art. 258). A closer analysis shows that exact definitions of both force and mass as measurable quantities are supplied by the laws of motion.

15

Notice first that if it is possible to subject two different bodies to the action of equal forces, the second law furnishes a definite comparison of their masses. Let bodies whose masses are m_1 and m_2 be acted upon by equal forces of magnitude P, and let p_1, p_2 be their accelerations. Then (Art. 253)

$$m_1 p_1 = m_2 p_2;$$

and if p_1 and p_2 can be measured, the ratio of the masses is at once known. If a third mass m_3 be acted upon by a force equal to P, the ratio of its mass to that of m_1 or m_2 becomes known if its acceleration p_3 is measured. In fact,

$$m_2 = (p_1/p_2)m_1; \quad m_3 = (p_1/p_3)m_1;$$

and if m_1 is chosen as the unit mass, the numerical values of m_2 and m_3 are known. Thus the mass of any body whatever can be expressed in terms of m_1 as a unit if it can be determined what acceleration it will have when acted upon by a force equal to P.

Evidently, however, it is not necessary always to apply a force of the same magnitude P. The masses of two particles may be compared by comparing the accelerations produced in them by equal forces of any magnitude whatever; and two masses may be compared indirectly, by comparing each independently with a third mass, forces of different magnitudes being used in the two comparisons.

Now our conception of the laws of motion implies that all such comparisons of two masses m_1, m_2, direct and indirect, with any values whatever of the forces whose effects are measured in making the comparisons, lead to the same value of the ratio m_1/m_2. This must be true if the science based upon these laws is self-consistent. Assuming it to be true, we have a method of defining mass as a measurable magnitude in a consistent and exact manner.

But is it possible to define the meaning of "equal forces applied to different bodies" without making use of the notion of mass itself? The answer to this question is that such a definition is supplied by the third law,—the equality of action and reaction.*

---

* Maxwell's answer is that the desired object may be accomplished by conceiving the forces applied by means of an elastic string, one end of which is attached to each of two or more bodies in turn. The other end is pulled in such a way that the string elongates by a certain amount which is the same in every case and is kept constant during each experiment. Assuming the

The acceleration of a particle at any instant is due to the influence of other particles. The "action" of a particle A upon a particle B which causes acceleration of B is called the "force exerted by A upon B." The third law of motion asserts that A and B influence each other *mutually*, so that the force exerted by A upon B and the force exerted by B upon A are at every instant equal in magnitude and opposite in direction.

Let the masses of the two particles be m_1 and m_2. Let that part of the acceleration of A which is due to B be p_1, and let that part of the acceleration of B which is due to A be p_2. The "force" exerted by B upon A is then measured by $m_1 p_1$, and the force exerted by A upon B by $m_2 p_2$. By the third law, these are equal in magnitude; that is,

$$m_1 p_1 = m_2 p_2, \quad \text{or} \quad m_2/m_1 = p_1/p_2.$$

This equation may be regarded as the definition of the ratio of the masses of the two bodies. Or, in words,

The masses of any two bodies are in the inverse ratio of the accelerations they give each other at any instant.

This conception of the meaning of mass and of force, and of the significance of the laws of motion, may be embodied in the following statements:

(1) The acceleration of a particle is always due to the influence of other particles.

(2) The actual acceleration of a particle is at every instant equal to the vector sum of accelerations due to the individual particles which are influencing it.

(3) The accelerations which two particles experience by reason of their mutual influence are at every instant oppositely directed along the line joining them.

(4) The masses of two particles are in the inverse ratio of the accelerations they give each other. This is a definition of mass.

(5) The product of the mass of a particle A into that part of its acceleration which is due to another particle B is the measure of the force exerted by B upon A. This is a definition of force.

---

elastic properties of the string to remain constant, it may be assumed that the magnitude of the force is the same in every case. ("Matter and Motion," Chapter III.) Even if no actual string can be found for which this will be true, this may perhaps still be regarded as a satisfactory method of explaining *ideally* what is meant by equal forces.

The first of these statements is virtually Newton's first law; for it implies that if a particle were free from the influence of other particles (that is, from the action of forces) it would have no acceleration.

The second statement is the law of composition of forces, usually regarded as included in Newton's second law. The second law as applied to the effect of a single force is included in statements (4) and (5). Newton's third law is included in (3), (4) and (5).*

261. **Particles and Bodies.**—The geometrical analysis of motion in the present Chapter and in Chapter XII referred to a particle. The laws of motion, although stated by Newton as applying to "bodies," have here been applied only to the ideal bodies called particles. The propositions at the end of Art. 260 were also stated as applying to particles. These particles have been regarded as possessing finite mass while occupying no finite volume. That such a particle exists, either in isolation or associated with other particles so as to constitute one of the "bodies" of our experience, we are not justified in asserting. It is, then, pertinent to inquire what applicability this abstract theory has to the actual bodies of nature.

To answer this question it is necessary to anticipate the results which are reached when the theory of the motion of a particle is extended to systems of connected particles. Assuming the acceleration of every particle to be in accordance with Newton's second law as above explained, and assuming the influence of the particles upon one another to be in accordance with Newton's third law, it is possible to deduce certain general laws governing the motion of connected systems of particles. These laws are found to describe with great accuracy the actual motions of natural bodies.

Moreover, it is immaterial whether a body is regarded as made up of a finite number of particles, each of finite mass, or whether it is regarded as occupying space continuously (Art. 5). If by particle we understand a portion of matter whose volume and mass are both vanishingly small, the foregoing theory of the motion of a particle may be applied if it be assumed that the force acting upon any particle becomes vanishingly small with the mass. To extend the theory

---

* For a critical analysis of the laws of motion along the lines of the above discussion, the following works may be consulted: "Science of Mechanics," by Dr. Ernst Mach, English translation by Thomas J. McCormack. "Grammar of Science," by Karl Pearson. "Theoretical Mechanics," by A. E. H. Love.

to the motion of a body of finite size involves, on this hypothesis, a process of integration, but the general laws deduced do not differ from those resulting from the other assumption.

Newton's first and second laws are not intelligible as applied to a body of finite size, unless it be assumed that all parts of the body have at every instant equal and parallel velocities. This is, indeed, a possible state of motion, but its actual occurrence is very exceptional. It is not possible to describe simply the way in which a body would move if not influenced by other bodies ; still less can a simple general statement be made describing the effect of a single external force in accelerating the various parts of a body.

One of the results reached in the theory of the motion of connected systems of particles is that there is in every such system a certain point (the center of mass) whose motion is accurately described by Newton's first and second laws, if the whole mass be regarded as concentrated at that point and acted upon by forces equal in magnitude and direction to the external forces which are actually applied to the system.

§ 4. *Simultaneous Motions.*

262. Meaning of Simultaneous Motions.— The actual motion of a particle is sometimes regarded as the resultant of two or more component motions occurring *simultaneously*.

Thus, if the total displacement during a given interval is AB (Fig. 119), the particle may be regarded as receiving simultaneously the displacements AC, CB, even though the actual displacement follows a path AMB not containing the point C.

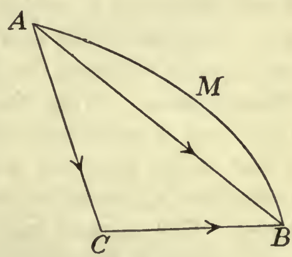


FIG. 119.

Similarly, if the actual velocity at a given instant is represented by the vector $A'B'$ (Fig. 120), the particle may be regarded as having simultaneously two velocities represented by $A'C'$, $C'B'$. And if the actual acceleration at any instant is represented by the vector $A''B''$ (Fig. 121), the particle may be regarded as having simultaneously two accelerations represented by $A''C''$, $C''B''$.

Any actual displacement, velocity or acceleration may thus be regarded as the resultant of any number of "simultaneous" components which satisfy the condition that their vector sum is equal to the given resultant.

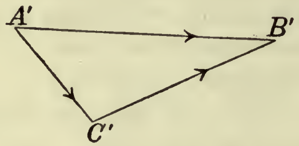


FIG. 120.

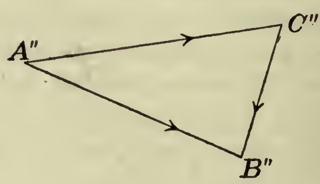


FIG. 121.

263. Resolution Into Simultaneous Motions an Arbitrary Process.— Obviously a particle cannot at the same time describe two different paths; neither can it have, at the same instant, two different velocities or accelerations.

The motion has at every instant a definite direction and a definite speed, and the actual velocity is represented by a definite vector. This may be expressed as the sum of several component vectors, but it is only by an arbitrary use of language that these component vectors can be said to represent velocities actually possessed by the particle at the same instant.

264. Effects of Simultaneous Forces May Be Estimated Separately.— One case in which it may be advantageous to regard the actual motion of a particle as the resultant of two or more simultaneous motions is that in which the particle is acted upon by two or more forces such that the effect of each acting alone admits of simple determination. From the law of composition of forces (Art. 255) the actual acceleration is the vector sum of the accelerations which would result from the several forces acting singly. Hence these component accelerations may be computed separately and then combined. This method may be carried further, and applied to the computation of the velocity-increment and the displacement due to the simultaneous action of several forces, constant or variable, during any interval.

265. Velocity-Increment Due to Several Forces Acting Simultaneously.— Since acceleration is by definition (Art. 247) the increment of velocity per unit time, and since, by the law of composition of forces, the acceleration at each instant is the vector sum of the

accelerations which would result from the separate action of the forces, it follows immediately that *the actual increment of velocity produced in any interval is the vector sum of the velocity-increments which would result from the separate action of the forces during the same interval.*

If at the beginning of a certain interval the velocity is v_0, and if during the interval the particle is acted upon by forces P_1, P_2, . . . , constant or variable, which separately would produce velocity-increments v_1, v_2, . . . , the velocity at the end of the interval is the vector sum of v_0, v_1, v_2, ...

266. Displacement Due to Several Forces Acting Simultaneously.— Since the displacement of a particle during any interval depends upon the magnitude and direction of its velocity at every instant throughout that interval, the total displacement may be computed as the vector sum of the displacements due to any components into which the actual velocity may be resolved. Thus, suppose a particle, having at the beginning of a certain interval Δt some definite velocity v_0, to be acted upon during the interval by a force P. The position of the particle at the end of the interval may be determined as follows:

Determine separately (a) the displacement in the time Δt due to the initial velocity if no force acted, and (b) the displacement in the time Δt due to the force P acting on the particle initially at rest. The vector sum of these displacements is the actual resultant displacement. This vector, drawn from the initial position of the particle, determines its position at the end of the interval.

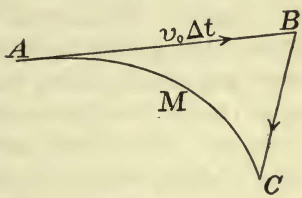


FIG. 122.

Thus, the displacement due to the initial velocity v_0 is a vector whose direction is that of v_0 and whose length is $v_0 \Delta t$. Represent this by AB (Fig. 122). If BC represents the displacement which the particle would receive in the time Δt if initially at rest and acted upon by the force P, the vector AC represents the resultant displacement.

The path followed by the particle from A to C is some curve AMC tangent to AB at A.

If two forces P_1 and P_2 act upon the particle during the interval Δt, the total displacement may be computed as the vector sum of

three components: a component $v_0 \Delta t$ having the direction of the initial velocity v_0, a component equal to the displacement of the particle if initially at rest and acted upon by the force P_1 alone, and a component equal to the displacement due to P_2 acting alone on the particle initially at rest.

The same method may be applied, whatever the number of the forces.

Examples.

1. A particle is projected horizontally with a velocity of 20 ft.-per-sec., after which it is acted upon by no force except gravity. Determine its velocity and its position after 0.1 sec., 0.2 sec., 0.5 sec., 2 sec.

Ans. Let v = velocity, ϕ = angle between v and the horizontal, r = distance from starting point, θ = angle between r and the horizontal. At the end of 2 sec. $v = 67.4$ ft.-per-sec., $\phi = 72° 45'$, $r = 75.8$ ft., $\theta = 58° 9'$.

2. A particle is projected horizontally with a velocity v_0, after which it is acted upon by no force except gravity. Determine its velocity and its position after t seconds.

3. A particle is projected with a velocity of 100 ft.-per-sec. in a direction inclined 20° upward from the horizontal. Determine its velocity and its position after 0.1 sec., 0.5 sec., 2 sec.

267. Relativity of Motion.— The motion of a body is always estimated with reference to some standard body which is assumed to be "fixed." Thus, in all ordinary practical problems, the earth is regarded as a fixed body, of unchanging dimensions; the displacement and velocity of any terrestrial body are estimated as they appear to an observer on the earth. If a body moves from A to B, these being two points fixed upon the earth, the straight line AB is its total displacement when the earth is the standard body with reference to which motions are specified. If, however, the rotation of the earth upon its axis is considered, the displacement has a very different value. Similarly, the velocity and acceleration of a particle at any instant have different values if different reference bodies are selected.

In the foregoing discussions, and in those that follow unless the contrary is explicitly stated, it is to be understood that only a single reference body is considered. When the motions of different bodies or particles are discussed, or when the motion of a particle is regarded as made up of components which follow certain rules of combination,

all these motions are estimated with reference to the same standard body.

268. Motion of a Particle Referred to Two Different Bodies. — In some cases the study of the motion of a particle is aided by regarding it from two different points of view, — the motion being referred first to one standard body and then to another. This is illustrated by the following simple case:

Let a particle P slide along a tube AB (Fig. 123), while the tube itself is in motion. Let the particle move from A to B while the tube moves from the position $A'B'$ to the position $A''B''$; AB being in every position parallel to its original position $A'B'$, and the ends of the tube describing the straight lines $A'A''$, $B'B''$. The total displacement of the particle is from A' to B''. The *actual* displacement may follow any curve or broken line lying between $A'B'B''$ and $A'A''B''$. If the motion of the particle along the tube from A to B, and the motion of the tube from $A'B'$ to $A''B''$, both take place uniformly, the actual displacement will be along the vector $A'B''$; but the same total displacement of the particle may occur in many other ways.

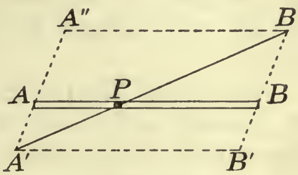


FIG. 123.

It will be observed that in thus analyzing the motion of P two different reference bodies are employed in estimating the motion. Thus, AB (or $A'B'$ or $A''B''$) is the displacement *of the particle with respect to the tube;* while $A'A''$ (or $B'B''$) is the displacement *of the tube with respect to some "fixed" body not specified;* and $A'B''$ is the displacement *of the particle with respect to this same "fixed" body.* Of these three displacements the third is seen to be equal to the vector sum of the first and second, and the actual displacement of the particle with respect to the "fixed" reference body is therefore regarded as the *resultant* of the two "simultaneous" displacements $A'B'$, $B'B''$.

Let it now be assumed that the particle P moves along the tube AB at a uniform rate, while the tube moves in the direction $A'A''$ also at a uniform rate. If the particle is at the point A of the tube when the tube is at $A'B'$, and if it moves from A to B while the tube moves from $A'B'$ to $A''B''$, the actual path of the particle is the vector $A'B''$. If this total motion occupies Δt seconds,

the actual velocity of the particle throughout the motion has the value

$$(\text{vector } A'B'')/\Delta t.$$

This is obviously the vector sum of two components,

$$(\text{vector } AB)/\Delta t \quad \text{and} \quad (\text{vector } B'B'')/\Delta t,$$

of which the former represents *the velocity of the particle with respect to the tube*, and the latter *the velocity of the tube with respect to the "fixed" reference body.*

Hence the "actual" velocity of the particle (*i. e.*, its velocity with respect to the assumed "fixed" reference body) may be regarded as the resultant of two simultaneous velocities, one of which is its velocity with respect to the tube and the other is the velocity of the tube.

An analysis like the above is useful in many cases in which the displacement and velocity of a particle may easily be estimated with reference to a body which is itself in motion.

Examples.

1. A straight tube AB, 1 met. long, moves in such a way that every point has a velocity, relative to the earth, of 12 c.m.-per-sec. in a direction inclined 45° to AB. A particle slides uniformly from A to B in 3 sec. Required (*a*) the velocity of the particle relative to the tube; (*b*) its velocity relative to the earth; (*c*) its total displacement relative to the earth while it slides from A to B. [Give direction as well as magnitude of each of the required vector quantities.]

2. A stream of water flows uniformly at the rate of 2 miles per hour. A boat is rowed in such a way that in still water its velocity would be 5 ft.-per-sec. in a straight line. If headed directly across the current, (*a*) what is its velocity referred to the earth? (*b*) If the stream is 3,000 ft. wide and the boat starts from one shore, where will it strike the opposite shore?

3. If, in Ex. 2, the boat is headed up-stream at an angle of 30° with the shore, answer questions (*a*) and (*b*).

Ans. (*a*) 2.86 ft.-per-sec., 29° 12′ up-stream. (*b*) 0.317 miles up-stream.

4. In Ex. 2, how must the boat be headed in order to strike the opposite shore as near as possible to the starting point?

Ans. 35° 55′ up-stream.

5. Show that the current does not affect the time of crossing the stream.

6. A straight tube AB, 4 ft. long, rotates about the end A at the uniform rate of 1 revolution per sec. A particle slides along the tube from B toward A at the rate of 10 ft.-per-sec. Determine the velocity of the particle relative to the earth when it is midway between A and B.

Ans. If C is the position of the particle, its velocity is 16.06 ft.-per-sec. in a direction making the angle $51° \; 30'$ with CA.

CHAPTER XV.

PLANE MOTION OF A PARTICLE.

§ 1. *Methods of Specifying Motion in a Plane.*

269. **Restriction to Plane Motion.**— The principles developed in Chapter XIV apply to any motion of a particle in space. In considering the methods of applying these principles to the solution of particular problems, the discussion will be limited mainly to the case in which the path of the particle lies in a plane. It is needful now to consider how the motion can conveniently be specified in this restricted case.

270. **Coördinates of Position.**— If the motion of a particle is restricted to a plane, its *position* at any instant may be specified by the values of two quantities. Any two quantities serving this purpose are *coördinates of position.*

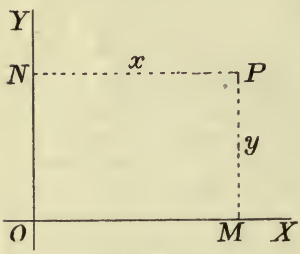


FIG. 124.

Of the various possible systems of coördinates, the most useful are the *rectangular* system and the *polar* system.

Rectangular coördinates.—Let OX and OY (Fig. 124) be any two fixed lines at right angles to each other, called axes of coördinates, and let P be the position of the moving particle at any instant. Then the distances PN and PM, measured parallel to OX and OY respectively, are the *rectangular coördinates* of P, and will be denoted by x and y.

As the particle moves, x and y vary. If x and y are known functions of the time, the position of P is known at every instant.

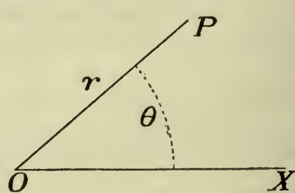


FIG. 125.

Polar coördinates. — Let the fixed pcint O and the fixed line OX (Fig. 125) be taken as *pole* and *initial line* respectively; then the angle POX (or θ) and the length OP (or r) are the *polar coördinates* of the particle P. As P moves,

its position is known at every instant if r and θ are known functions of the time.

271. Velocity and Acceleration Each Given by Two Components.— Any vector quantity lying in a given plane is known when its components in two given directions are specified. In analyzing the motion of a particle in a plane, the displacement, velocity and acceleration may each be regarded as the resultant (or vector sum) of two components, the directions of resolution being chosen at pleasure. The particular directions chosen will be determined by convenience, depending upon the nature of the problem under consideration. In nearly every case, however, the two directions of resolution will be taken at right angles to each other.

The values of the two components of the velocity and of the acceleration may be expressed in terms of the coördinates of position and their time-derivatives.

272. Resolution Parallel to Coördinate Axes.— If the position of the particle is specified by its rectangular coördinates, it is usually convenient to resolve the displacement, velocity and acceleration parallel to the coördinate axes. The components in these directions may be called *axial components*.

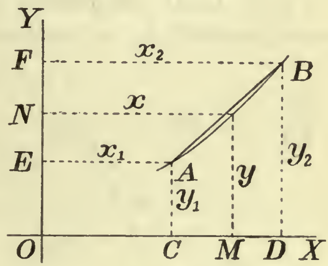


FIG. 126.

273. Axial Components of Displacement. — If the particle moves from A to B (Fig. 126) during any interval Δt, its total displacement is given by the vector AB. This total displacement may be resolved into components CD and EF, parallel to the coördinate axes. Let x, y be the coördinates of position at any instant; let x_1, y_1 be the coördinates of A, and x_2, y_2 those of B. Then the axial components of the displacement are evidently equal to $x_2 - x_1$ and $y_2 - y_1$.

274. Axial Components of Velocity.— The average velocity of the particle during the interval Δt is (vector AB)/Δt; the exact value of the velocity at the beginning of the interval is the limit approached by this average value as Δt is made to approach zero (Art. 244). But since $x_2 - x_1$ and $y_2 - y_1$ are always the values of the axial com-

ponents of vector AB, the axial components of limit [(vector AB)/Δt] are

$$\text{limit } [(x_2 - x_1)/\Delta t] \quad \text{and} \quad \text{limit } [(y_2 - y_1)/\Delta t].$$

The axial components of the velocity are therefore

$$\begin{aligned} \text{limit } [(x_2 - x_1)/\Delta t] &= \text{limit } [\Delta x/\Delta t] = dx/dt = \dot{x} \\ &= x\text{-component}; \\ \text{limit } [(y_2 - y_1)/\Delta t] &= \text{limit } [\Delta y/\Delta t] = dy/dt = \dot{y} \\ &= y\text{-component}. \end{aligned}$$

Let s denote the length of the path measured from some fixed point up to the position of the particle; v the magnitude of the resultant velocity; α, β the angles between v and the x- and y-axes respectively. Then

$$\begin{aligned} v &= ds/dt; \\ \dot{x} &= dx/dt = v \cos \alpha = (ds/dt) \cos \alpha; \\ \dot{y} &= dy/dt = v \cos \beta = (ds/dt) \cos \beta; \\ v^2 &= \dot{x}^2 + \dot{y}^2; \\ \tan \alpha &= \text{cotan } \beta = \dot{y}/\dot{x}. \end{aligned}$$

275. Axial Components of Velocity-Increment.— Let velocities be represented by vectors laid off from a point as in Art. 245. Let O' (Fig. 127) be this point, and from O' draw $O'X'$ and $O'Y'$ parallel respectively to the axes of x and y. Let vectors $O'A'$, $O'B'$ represent the values of the velocity at the beginning and end of an interval Δt. The velocity-increment for the interval is then represented by vector $A'B'$. This increment is completely known if we know its axial components $C'D'$ and $E'F'$.

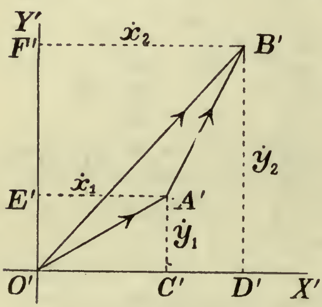


FIG. 127.

As found above, $\dot{x}$ and $\dot{y}$ are the values of the axial components of the velocity at any instant. Let $\dot{x}_1$, $\dot{y}_1$ be the values of these components at the beginning of the interval, and $\dot{x}_2$, $\dot{y}_2$ their values at the end of the interval. Then (Fig. 127)

$$\dot{x}_1 = O'C', \quad \dot{y}_1 = O'E'; \quad \dot{x}_2 = O'D', \quad \dot{y}_2 = O'F'.$$

Hence $\qquad C'D' = \dot{x}_2 - \dot{x}_1; \quad E'F' = \dot{y}_2 - \dot{y}_1.$

The axial components of the velocity-increment are therefore

$$\dot{x}_2 - \dot{x}_1 = \Delta\dot{x} = x\text{-component};$$
$$\dot{y}_2 - \dot{y}_1 = \Delta\dot{y} = y\text{-component}.$$

276. Axial Components of Acceleration.— The average acceleration for the interval Δt is equal to (vector $A'B'$)/Δt. The exact value of the acceleration at the beginning of the interval is the limit approached by (vector $A'B'$)/Δt as Δt is made to approach zero (Art. 250). Since the axial components of vector $A'B'$ are equal to $\dot{x}_2 - \dot{x}_1$ and $\dot{y}_2 - \dot{y}_1$, or $\Delta\dot{x}$ and $\Delta\dot{y}$, whatever the length of the interval, the axial components of limit [(vector $A'B'$)/Δt] are

$$\text{limit } [\Delta\dot{x}/\Delta t] \quad \text{and} \quad \text{limit } [\Delta\dot{y}/\Delta t].$$

That is, the axial components of the acceleration are

$$\text{limit } [\Delta\dot{x}/\Delta t] = d\dot{x}/dt = d^2x/dt^2 = \ddot{x} = x\text{-component};$$
$$\text{limit } [\Delta\dot{y}/\Delta t] = d\dot{y}/dt = d^2y/dt^2 = \ddot{y} = y\text{-component}.$$

Let p denote the actual or resultant acceleration at any instant, and α', β' the angles its direction makes with the x- and y-axes respectively. Then

$$\ddot{x} = d^2x/dt^2 = p \cos \alpha';$$
$$\ddot{y} = d^2y/dt^2 = p \cos \beta';$$
$$p^2 = \ddot{x}^2 + \ddot{y}^2;$$
$$\tan \alpha' = \text{cotan } \beta' = \ddot{y}/\ddot{x}.$$

277. Hodograph.— If the vector representing the velocity at any instant is $O'P'$ (Fig. 128), O' being a fixed point while P' moves as the value of the vector varies, the curve traced by P' is the *hodograph* (Art. 245).

The rectangular coördinates of the point P', referred to axes $O'X'$, $O'Y'$, parallel to the axes of x and y, are $\dot{x}$ and $\dot{y}$. If the values of $\dot{x}$ and $\dot{y}$ are known at every instant, the form of the curve can be determined.

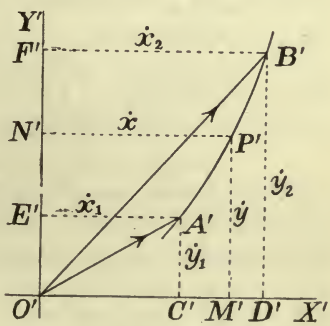


FIG. 128.

278. Cases Adapted to Polar Coördinates.—There are certain problems to whose treatment polar coördinates are well adapted. This is often the case when the acceleration of the particle is directed toward a fixed point. In this case, and in some others, it is convenient to replace the acceleration and the velocity by their resolved parts in directions parallel and perpendicular to the radius vector. The values of these components in terms of r and θ and their derivatives will now be deduced.

279. Components of Velocity Parallel and Perpendicular to Radius Vector.— Let a particle move from A to B along the curve AB (Fig. 129) during the interval Δt. Let r, θ be its polar coördinates at any time t, and let the initial values of these coördinates be r_1, θ_1, and the final values r_2, θ_2. Then $r_1 = OA$, $r_2 = OB$, $\theta_1 = AOX$, $\theta_2 = BOX$.

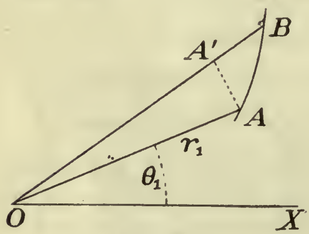


FIG. 129.

The average velocity during the interval is (vector AB)/Δt, and the true velocity at the beginning of the interval is the limit of this value as Δt is made to approach zero. Take $OA' = OA = r_1$, and let (vector AB) be resolved into components (vector AA') and (vector $A'B$). Then the true velocity at the beginning of the interval is equivalent to two components

$$\text{limit }[(\text{vector } AA')/\Delta t], \quad \text{limit }[(\text{vector } A'B)/\Delta t].$$

The latter component has the direction OA and the former is perpendicular to OA. Hence the component of velocity parallel to the radius vector is

$$\text{limit }[A'B/\Delta t] = \text{limit }[(r_2 - r_1)/\Delta t] = \text{limit }[\Delta r/\Delta t] = dr/dt,$$

and the component of velocity perpendicular to the radius vector is

$$\begin{aligned} \text{limit }[AA'/\Delta t] &= \text{limit }[r_1(\theta_2 - \theta_1)/\Delta t] \\ &= \text{limit }[r_1 \Delta\theta/\Delta t] = r(d\theta/dt). \end{aligned}$$

These results may also be deduced from the values of the axial components of velocity. Thus, let x, y be the coördinates of position referred to a pair of rectangular axes, the origin being at the pole and the x-axis coinciding with the initial line of the polar coördinates r, θ. (Fig. 130.) Then

$$x = r\cos\theta; \quad y = r\sin\theta. \quad . \quad . \quad . \quad (1)$$

The velocity is equivalent to a component $\dot{x}$ in direction OX and a component $\dot{y}$ in direction OY. The resolved part of the velocity in any direction is equal to the sum of the resolved parts of $\dot{x}$ and $\dot{y}$ in that direction. Hence the resolved parts of the velocity along and perpendicular to OP are as follows:

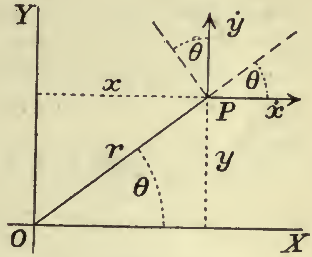


FIG. 130.

$\dot{y} \sin \theta + \dot{x} \cos \theta =$ component along OP;

$\dot{y} \cos \theta - \dot{x} \sin \theta =$ component perpendicular to OP.

The values of $\dot{x}$ and $\dot{y}$ may be expressed in terms of r, θ and their derivatives by differentiating equations (1) with respect to t. Thus,

$$\left.\begin{aligned} \dot{x} &= \frac{dr}{dt} \cos \theta - r \sin \theta \frac{d\theta}{dt}; \\ \dot{y} &= \frac{dr}{dt} \sin \theta + r \cos \theta \frac{d\theta}{dt}. \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

Substituting these values, the components of velocity parallel and perpendicular to the radius vector become

$$\frac{dr}{dt} = \text{component in direction } OP;$$

$$r\frac{d\theta}{dt} = \text{component perpendicular to } OP.$$

280. Components of Acceleration Parallel and Perpendicular to Radius Vector.— Since the acceleration is equivalent to two components, $\ddot{x}$ in direction OX (Fig. 131) and $\ddot{y}$ in direction OY, its resolved parts parallel and perpendicular to OP are as follows:

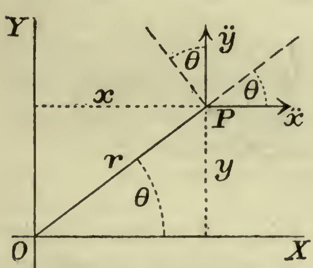


FIG. 131.

$\ddot{y} \sin \theta + \ddot{x} \cos \theta$ in direction OP;

$\ddot{y} \cos \theta - \ddot{x} \sin \theta$ perpend. to OP.

The values of $\ddot{x}$ and $\ddot{y}$ in terms of r, θ and their derivatives may be found

by differentiating equations (2), Art. 279, with respect to t. The resulting values are

$$\ddot{x} = \frac{d^2r}{dt^2}\cos\theta - 2\frac{dr}{dt}\frac{d\theta}{dt}\sin\theta - r\left(\frac{d\theta}{dt}\right)^2\cos\theta - r\frac{d^2\theta}{dt^2}\sin\theta;$$

$$\ddot{y} = \frac{d^2r}{dt^2}\sin\theta + 2\frac{dr}{dt}\frac{d\theta}{dt}\cos\theta - r\left(\frac{d\theta}{dt}\right)^2\sin\theta + r\frac{d^2\theta}{dt^2}\cos\theta.$$

Hence $$\ddot{y}\sin\theta + \ddot{x}\cos\theta = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$=$ component of acceleration parallel to radius vector;

$$\ddot{y}\cos\theta - \ddot{x}\sin\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$=$ component of acceleration perpendicular to radius vector.

The value of the latter component may be written in another form, as follows:

$$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

This may be verified by differentiation.

281. Angular Motion of a Particle.—The line joining a moving particle A to a fixed point O in general turns about O as the particle moves. Let θ denote the angle between OA and a fixed line in the plane of the motion; then the rate of turning of the line is measured by the rate of change of θ. By the "angular motion of A about O" is meant the angular motion of the line OA.

Angular velocity may be defined as the *rate of angular motion.* It is measured by the angle described per unit time. If the line OA turns at a uniform rate, let $\Delta\theta$ be the increment of θ in a time Δt, and let ω denote the angular velocity; then

$$\omega = \Delta\theta/\Delta t.$$

If θ increases at a variable rate, $\Delta\theta/\Delta t$ is the *average angular velocity* for the time Δt, and the instantaneous value of the angular velocity is the limit of $\Delta\theta/\Delta t$ as Δt approaches 0; that is,

$$\omega = d\theta/dt.$$

Angular acceleration is the *rate of change of angular velocity.* If the angular velocity ω varies at a uniform rate, let $\Delta\omega$ be its incre-

ment during a time Δt, and let ϕ denote the angular acceleration. Then

$$\phi = \Delta\omega/\Delta t.$$

If ω changes at a variable rate, $\Delta\omega/\Delta t$ is the *average angular acceleration* for the time Δt, and the instantaneous value of the angular acceleration is the limit of $\Delta\omega/\Delta t$ as Δt approaches 0; that is

$$\phi = d\omega/dt.$$

If the point A is describing a circle of radius r with center at O, and if Δs is the length of arc subtended by the angle $\Delta\theta$,

$$\Delta s = r\Delta\theta;$$

hence

$$ds/dt = r(d\theta/dt).$$

That is, the linear velocity is at every instant equal to the product of the radius by the angular velocity. The angle θ must be expressed in radians in order that this relation may hold.

EXAMPLES.

1. A particle describes a circle in any manner. Show that its angular velocity about any point of the circumference is at every instant equal to half its angular velocity about the center. The angular velocities about all points in the circumference are equal.

2. A particle P describes a straight line at the uniform rate of 24 ft.-per-sec. Determine its angular velocity and angular acceleration about a point A, 6 ft. from the path, in two positions: (*a*) when AP is at right angles to the path, (*b*) when AP is inclined 45° to the path.

Ans. (*a*) $\omega = 4$ rad.-per-sec., $\phi = 0$. (*b*) $\omega = 2$ rad.-per-sec., $\phi = 8$ rad.-per-sec.-per-sec.

3. A particle P describes a straight line in such a way that its angular velocity about a point A distant h from the path is constant and equal to ω. Determine the linear velocity and linear acceleration in any position.

Ans. Let θ denote the angle between AP and a perpendicular to the path, v the linear velocity and p the linear acceleration; then $v = h\omega \sec^2\theta$, $p = 2h\omega^2 \tan\theta \sec^2\theta$.

282. Relation of Velocity and Acceleration to Path.—The methods of resolution above considered, though useful in the analysis of particular cases of motion, do not show clearly the relation of velocity and acceleration to the path of the particle. This relation is best shown by resolving along the tangent and normal to the path.

The values of the components of velocity and acceleration thus determined are independent of any particular system of coördinates of position.

283. Tangential and Normal Components of Velocity.— Since the resultant velocity, for any position of the particle, has the direction of the tangent to the path, its resolved part in the direction of the tangent is the velocity itself, its value being ds/dt (Art. 244); while the resolved part along the normal is zero.

284. Tangential and Normal Components of Acceleration.— Let A (Fig. 132) be the position of the particle at an instant t, and AB the path described during an interval Δt. The speed in the position A will be denoted by v.

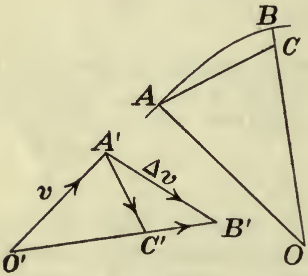


FIG. 132.

From any point O' lay off the vector $O'A'$ to represent the initial velocity and $O'B'$ to represent the final velocity; these vectors are then parallel to the tangents to the path at A and at B respectively.

The velocity-increment during the interval is equal to (vector $A'B'$); the average acceleration is (vector $A'B'$)$/\Delta t$; and the true acceleration at the beginning of the interval is the limit of this average value as Δt is made to approach zero. The vector $A'B'$ will now be replaced by two components whose limiting directions are those of the tangent and normal to the path at A. These components are $A'C'$, $C'B'$, if the point C' is so taken that $O'C' = O'A' = v$. It is evident that as Δt approaches zero, the limiting direction of $A'C'$ is perpendicular to $O'A'$ and that of $C'B'$ is parallel to $O'A'$; but $O'A'$ is parallel to the tangent to AB at A.

The tangential and normal components of the acceleration at the instant t are therefore

$$\text{limit } [(\text{vector } C'B')/\Delta t] \quad \text{and} \quad \text{limit } [(\text{vector } A'C')/\Delta t].$$

It remains to determine the magnitudes of these limiting values.

Evidently $C'B'$ is equal in magnitude to the increment received by the *speed* during the interval Δt. Hence

$$\text{limit } [C'B'/\Delta t] = \text{limit } [\Delta v/\Delta t] = dv/dt,$$

which is the value of the tangential component of the acceleration. Next, to determine the normal component.

Draw normals to the path at A and B, and let O be their point of intersection. Lay off $OC = OA$. The triangles AOC, $A'O'C'$ are similar; hence

$$A'C'/A'O' = AC/AO;$$

$$A'C' = AC(A'O'/AO) = AC(v/AO);$$

$$\text{limit } [A'C'/\Delta t] = \text{limit } [(v/AO)(AC/\Delta t)].$$

As Δt approaches zero, the point O approaches as a limiting position the center of curvature of the curve AB at A, and the limiting value of AO is the radius of curvature at A. Call this R. Again, as Δt approaches zero, the points C and B approach coincidence; AC and arc AB approach a ratio of equality. Hence

$$\text{limit } [AC/\Delta t] = \text{limit } [(\text{arc } AB)/\Delta t] = \text{limit } [\Delta s/\Delta t] = ds/dt = v.$$

Substituting these values,

$$\text{limit } [A'C'/\Delta t] = (v/R)v = v^2/R.$$

The required components of the resultant acceleration p are therefore

$$dv/dt = d^2s/dt^2 = \text{tangential component};$$

$$v^2/R = (ds/dt)^2/R = \text{normal component}.$$

These values are independent of any particular system of coördinates of position, but they may, if desired, be expressed in terms of the coördinates.

Examples.

1. Determine the magnitude and direction of the resultant acceleration of a particle describing a circle of radius r with uniform speed v.

[Express the values of the tangential and normal components and determine their resultant. Compare with result of Ex. 4, following Art. 251.]

2. The path of a particle is a circle of radius r lying in a vertical plane. The speed is given by the equation $v = k\sqrt{z}$, where z is the vertical distance below a horizontal plane through the center of the circle. Compute the tangential and normal components of the acceleration for any value of z.

Ans. Tangential component $= (k^2/2r)\sqrt{(r^2 - z^2)}$; normal component $= k^2z/r$.

3. In Ex. 2, let $v = 8$ ft.-per-sec. when $z = 1$ ft., and let the radius of the circle be 2 ft. Compute the tangential and normal components of the acceleration when the particle is in its lowest position.

Ans. Tangential component $= 0$; normal component $= 64$ ft.-per-sec-per-sec.

4. A particle describes a circle of 50 ft. radius in such a way that $s = 16t^2$, in which s is in feet and t in seconds. Compare the true acceleration when $t = 2$ with the average acceleration for the ensuing 0.1 second.

§ 2. *Motion in a Plane Under Any Forces.*

285. Conditions Under Which the Motion Will Be Confined to a Plane.— If the direction of the resultant force acting upon a particle is always parallel to a fixed plane, and if the motion at any instant is parallel to that plane, the resolved part of the velocity perpendicular to the plane will remain zero, and the path of the particle will therefore lie in the plane.

286. Two Independent Equations of Motion.— The general equation of motion, $P = mp$, is a *vector equation*, expressing equality of direction as well as of magnitude between the two members. If both P and p be resolved in any direction, we have

$$(\text{resolved part of } P) = m \times (\text{resolved part of } p).$$

Hence by resolving in different directions, any number of true equations may be written. In the case of plane motion, however, only two of these equations can be independent. For if the resolved parts of a vector in two directions are given, the vector is completely determined; hence the two equations obtained by resolving in two different directions include all that is expressed by any equation obtained by resolving in a third direction.

The form of the two independent equations depends upon the system of coördinates employed in specifying the position of the particle, and also upon the directions of resolution.

287. Equations of Motion in Terms of Rectangular Coördinates.— Let the position of a particle be specified by its coördinates x and y, referred to any pair of fixed rectangular axes in the plane of the motion, and let the two equations of motion be obtained by resolving parallel to the axes. Let the axial components of the resultant force P be X and Y. If several forces act upon the particle, X

is the algebraic sum of their resolved parts in the x-direction, and Y the algebraic sum of their resolved parts in the y-direction. The axial components of the acceleration are $\ddot{x}$ and $\ddot{y}$ (Art. 276). The equations of motion are therefore

$$X = m\ddot{x} = m(d^2x/dt^2)\,; \quad . \quad . \quad . \quad (1)$$

$$Y = m\ddot{y} = m(d^2y/dt^2). \quad . \quad . \quad . \quad (2)$$

288. Equations of Motion in Terms of Polar Coördinates.— If polar coördinates are employed, the equations are obtained in the most convenient form by resolving parallel and perpendicular to the radius vector. Let P_r and P_θ be the resolved parts of the resultant force P in these directions respectively. The values of the resolved parts of the acceleration are given in Art. 280. Using these values, the equations of motion become

$$P_r = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$P_\theta = m\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right) = m\left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]. \quad . \quad (2)$$

289. Resolution Along Tangent and Normal to Path.— Let the components of the resultant force P resolved along the tangent and normal to the path at the instantaneous position of the particle be denoted by P_t and P_n respectively. Using the values of the resolved parts of the acceleration in these directions (Art. 284), the equations of motion become

$$P_t = m(dv/dt) = m(d^2s/dt^2)\,; \quad . \quad . \quad (1)$$

$$P_n = m(v^2/R). \quad . \quad . \quad . \quad . \quad (2)$$

290. Classes of Problems.— Problems relating to the motion of a particle in a plane may be classed as inverse or direct, according as their solution does or does not involve integration. (See Art. 223.) The most important cases fall under one of the following general problems:

(*a*) To determine the resultant force at any instant, the motion being completely known.

(*b*) To determine the motion when the forces are known.

The former problem is direct, the latter inverse.

291. Direct Problem: To Determine the Resultant Force When the Motion Is Known.— If the acceleration is known, the resultant force is given immediately by the general vector equation $P = mp$, or by any pair of algebraic equations equivalent to it. If the coördinates of position, or the components of the velocity, are known functions of the time, the components of the acceleration can be determined by differentiation.

EXAMPLES.

1. A body describes any curve at a uniform speed; determine the resultant force acting upon it.

2. A body of mass 12 lbs. describes the circumference of a circle 12 ft. in diameter at the uniform rate of 100 ft.-per-min. Required the resultant force acting upon it.

3. The position of a particle at any time is given by the equations $x = a + bt^2$, $y = A + Bt^3$. The mass being m, determine the magnitude and direction of the resultant force at the time t.

4. Determine the magnitude and direction of the resultant force at time t if the position is given by the equations $x = a + bt$, $y = A + Bt + Ct^2$.

5. What is the resultant force acting on the particle P in Ex. 3, Art. 281? If $h = 6$ ft., $\omega = 1$ rad.-per-sec., and the mass of the particle is 5 lbs., determine the value of the resultant force when the particle is 12 ft. from A. *Ans.* 415.7 poundals.

6. The path of a particle is an ellipse whose semi-axes are 20 ft. and 10 ft. The mass is 12 lbs. and the speed is uniformly 100 ft.-per-min. Determine the resultant force acting on the particle when at the extremity of each major axis.

292. Inverse Problem.— If the forces are known, the complete determination of the motion requires the integration of two simultaneous differential equations. These may be of various forms, depending upon the system of coördinates employed and also upon the directions of resolution. Usually it will be found convenient to use one of the three pairs of equations given in Arts. 287, 288, 289.

A problem of this class cannot be completely solved unless sufficient initial conditions are given for the determination of the constants of integration.

We proceed to the consideration of some important cases of plane motion.

§ 3. *Resultant Force Constant or Zero.*

293. Direction of Resultant Force Constant.— If the resultant force has a constant direction, let this be taken as the direction of the axis of x; then $X = P$, $Y = 0$; and the equations of motion (Art. 287) are

$$m(d^2x/dt^2) = P; \quad . \quad . \quad . \quad . \quad (1)$$

$$m(d^2y/dt^2) = 0. \quad . \quad . \quad . \quad . \quad (2)$$

Equation (1) cannot be integrated unless P is given. From (2),

$$dy/dt = C = \text{constant}; \quad . \quad . \quad . \quad (3)$$

$$y = Ct + C'. \quad . \quad . \quad . \quad . \quad (4)$$

Equation (3) expresses the fact that the y-component of the velocity remains constant. Motion of this character is often called *parabolic.*

The values of the constants C and C' may be determined if the values of y and dy/dt at some instant are known.

294. Resultant Force Constant in Direction and Magnitude.— If the resultant force is constant in magnitude as well as in direction, equation (1) can be integrated directly. Let

$$P/m = f = \text{constant};$$

then

$$d^2x/dt^2 = f.$$

Integrating twice,

$$dx/dt = ft + C''; \quad . \quad . \quad . \quad . \quad (5)$$

$$x = \tfrac{1}{2}ft^2 + C''t + C'''. \quad . \quad . \quad . \quad (6)$$

Equations (4) and (6) give the position at any time, and (3) and (5) the velocity at any time, as soon as the constants C, C', C'' and C''' are known. These may be found if the position and velocity at one instant are known.

295. Projectile.— A particle projected in any direction and left to the action of gravity and the resistance of the air is called a projectile. In the following discussion the resistance of the air is disregarded.

This problem is a special case of that treated in Art. 294. The solution there given applies, if the axes of coördinates are properly chosen. Obviously the axis of x must be vertical and the axis of y

horizontal, and g must replace f. In order to determine the constants of integration, let the following conditions be specified :

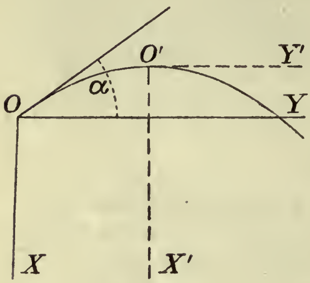


FIG. 133.

Let the particle be projected from a certain point which will be taken as origin of coördinates, the positive direction for x being downward. Let the velocity of projection be V, its direction making an angle α upward from the plus direction of the y-axis. Let t be reckoned from the instant when the particle is at the origin O (Fig. 133).

The equations expressing the values of the axial components of the velocity and of the coördinates of position are, as above found,

$$dx/dt = gt + C''; \quad dy/dt = C;$$

$$x = \tfrac{1}{2}gt^2 + C''t + C'''; \quad y = Ct + C'.$$

The initial conditions are that when $t = 0$,

$$x = 0, \quad y = 0, \quad dx/dt = -V \sin \alpha, \quad dy/dt = V \cos \alpha.$$

The values of the constants are therefore

$$C = V \cos \alpha, \quad C' = 0, \quad C'' = -V \sin \alpha, \quad C''' = 0,$$

and the equations become

$$dx/dt = gt - V \sin \alpha; \qquad (1)$$

$$dy/dt = V \cos \alpha; \qquad (2)$$

$$x = \tfrac{1}{2}gt^2 - V \sin \alpha \cdot t; \qquad (3)$$

$$y = V \cos \alpha \cdot t. \qquad (4)$$

Equations (1) and (2) give the velocity at any time, and equations (3) and (4) the position.

Equation of path.—Eliminating t between (3) and (4), the equation of the path is found to be

$$x = \frac{gy^2}{2V^2 \cos^2 \alpha} - y \tan \alpha. \qquad (5)$$

This is the equation of a parabola with its axis vertical.

The position of the vertex of the parabola may be found by so transforming coördinates that the equation takes the form $y^2 = 4mx$.

If we put $dx/dt = 0$ in (1) we find $t = (V \sin a)/g$; this value of t substituted in (3) and (4) gives

$$x = -(V^2 \sin^2 a)/2g; \quad y = (V^2 \sin a \cos a)/g.$$

These are the coördinates of the point at which the curve is horizontal, and this is the vertex of the parabola. For if, with this point as origin, the coördinates of any point of the curve are x', y', we have

$$x = x' - (V^2 \sin^2 a)/2g, \quad y = y' + (V^2 \sin a \cos a)/g.$$

These values substituted in (5) give

$$y'^2 = \frac{2V^2 \cos^2 a}{g} x',$$

which is the equation of a parabola whose vertex is at the origin O' and whose principal diameter is in the axis of x'.

Examples.

1. Determine the distance from O (Fig. 133) at which the body will cross the line OV. (This distance is called the *range* of the projectile.) *Ans.* $(V^2 \sin 2a)/g$.

2. Determine for what value of a the range is greatest. *Ans.* $a = 45°$.

3. Determine the greatest height to which the projectile will rise. *Ans.* $(V^2 \sin^2 a)/2g$.

4. If the body is projected with a velocity of 50 ft.-per-sec. in a direction inclined 25° upward from the horizontal, find (*a*) the equation of the path; (*b*) the position of the highest point reached with reference to the point of projection; (*c*) the range; and (*d*) the position and velocity at the end of 4 sec. from the instant of projection.

5. Prove that the same range will result from two different values of a. How are these two values related? *Ans.* They are complementary.

6. A particle is projected from a point A with a velocity V. What must be the direction of projection in order that it may pass through a point B such that the line AB is inclined at angle θ to the horizon and the distance $AB = a$?

Ans. If a = angle between V and horizontal, its value is given by the equation

$$\sin(2a - \theta) = \sin\theta + (ga \cos^2\theta)/V^2.$$

Show that the problem is impossible unless V^2 is at least as great as $ga(1 + \sin\theta)$.

7. A particle is to be projected from a given point in such a way as to pass through a point 50 ft. higher than the point of projection

and 200 ft. distant from it. What is the least allowable speed of projection? What is the corresponding direction of projection?

Ans. $V = 89.7$ ft.-per-sec.; $\alpha = 52° 14'$.

296. Resultant Force Zero.— If the resultant of all forces acting upon a particle is zero at any instant, the acceleration is also equal to zero. If the resultant force remains zero during any interval, the acceleration remains zero throughout that interval. The velocity therefore remains constant in magnitude and direction.*

297. Statics a Special Case of Kinetics.— Statics has been defined as that branch of Dynamics which treats of the conditions of equivalence of systems of forces, and especially of the conditions under which forces balance each other or are in equilibrium. (Art. 9.) The principles of Statics, although they may be developed to a great extent independently of the laws of motion (as shown in Part I), are also included in the principles of Kinetics.

Thus, in developing the laws of motion we have made use of the principle that the resultant of any system of concurrent forces is a force equal to their vector sum. The same principle was used in dealing with concurrent forces in Part I. All the methods of combining and resolving concurrent forces which have been developed in Statics are therefore included in Kinetics.

It will be noticed that the principle of the parallelogram (or triangle) of forces, which is the foundation of the methods of combining concurrent forces, has not been proved, but has been assumed as a fundamental axiom, both in the development of Statics (Art. 59) and in that of Kinetics (Art. 255).

298. Equilibrium.— A system of forces applied to a particle is in equilibrium if their combined action produces no effect upon the motion of the particle. The general condition of equilibrium for any set of forces is, therefore, that their resultant is zero. This does not imply that the particle is at rest; but that its acceleration is zero and its velocity uniform in magnitude and direction. This result is included in the meaning of the general equation of motion (Art. 256), as of course it should be.

---

*This is not given as a *proof* of the proposition that a particle acted upon by no forces, or by forces whose resultant is zero, moves in a straight line at a uniform rate. This principle has been assumed in the deduction of the general equation of motion, which must of course include the principle itself.

§ 4. *Central Force.*

299. Equations of Motion of Particle Acted Upon by Central Force.— A force which is always directed toward or from a fixed point is called a *central force.* The fixed point is a *center of attraction* or a *center of repulsion* according as the force is directed toward or from the center.

In many cases polar coördinates will be found most convenient in discussing the motion of a particle under a central force. We give the form taken by the differential equations, using both rectangular and polar coördinates.

Rectangular coördinates.— Let the center of force be taken as origin of coördinates, and let N (Fig. 134) be the position of the particle at the time t. If the resultant force is an attraction of magnitude P, we have

$$X = -P\cos\theta = -P(x/r);$$

$$Y = -P\sin\theta = -P(y/r).$$

The equations of motion are therefore

$$m(d^2x/dt^2) = -P(x/r); \quad . \quad . \quad . \quad (1)$$

$$m(d^2y/dt^2) = -P(y/r); \quad . \quad . \quad . \quad (2)$$

in which $r = \sqrt{x^2 + y^2}.$

Polar coördinates.— Take the center of force as pole, and choose any initial line. Resolving forces and accelerations parallel and perpendicular to the radius vector, the equations of motion given in Art. 288 become

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{P}{m}; \quad . \quad . \quad . \quad (3)$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0. \quad . \quad . \quad . \quad . \quad (4)$$

In the important case treated in the next Article it is convenient to employ rectangular coördinates. The subsequent discussion of central forces will refer mainly to the equations in polar coördinates and their application to particular cases.

300. Force Varying Directly as the Distance From a Fixed Point.—When the central force varies directly as the distance from

the fixed point, the equations in rectangular coördinates are readily integrated.

Let k denote the attraction per unit mass at unit distance from the center; then $P/m = kr$, and equations (1) and (2) of Art. 299 become

$$d^2x/dt^2 = -kx; \quad . \quad . \quad . \quad . \quad (5)$$

$$d^2y/dt^2 = -ky. \quad . \quad . \quad . \quad . \quad (6)$$

Each of these equations is identical in form with equation (1) of Art. 229, and they may be integrated separately by the method there employed. Four constants of integration will be introduced. In order to determine their values, let the following initial conditions be assumed:

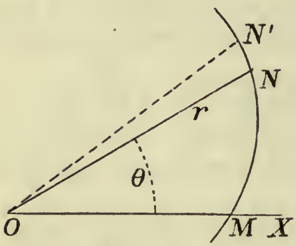


FIG. 134.

At a certain instant the particle is in the axis of x at a distance a from the origin, and is moving parallel to the axis of y with velocity V. Let t be reckoned from this instant. Then the initial conditions may be stated as follows:

When $t = 0$, $x = a$, $y = 0$, $\dot{x} = 0$, $\dot{y} = V$.

Integrating equation (5) as in Art. 229, and determining the constant from the condition that $\dot{x} = 0$ when $x = a$, there results

$$\dot{x}^2 = (dx/dt)^2 = k(a^2 - x^2). \quad . \quad . \quad . \quad (7)$$

Integrating again (Art. 229), and determining the constant from the condition that $x = a$ when $t = 0$,

$$k^{1/2}t = \sin^{-1}(x/a) - \sin^{-1}(1).$$

Taking cosine of each member, and multiplying by a, the result is

$$x = a \cos (k^{1/2}t). \quad . \quad . \quad . \quad . \quad (8)$$

Integrating equation (6) and determining the constant by the condition that $\dot{y} = V$ when $y = 0$,

$$\dot{y}^2 = (dy/dt)^2 = V^2 - ky^2 = k(b^2 - y^2), \quad . \quad . \quad (9)$$

if b^2 is written for V^2/k. A second integration gives

$$k^{1/2}t = \sin^{-1}(y/b) - \sin^{-1}(0),$$

the constant being determined by the condition that $y = 0$ when $t = 0$. Taking sine of each member,

$$\sin(k^{1/2}t) = \pm\, y/b.$$

The + sign must be used if the particle is moving in the positive y-direction when $t = 0$, because it is evident that y is + for small values of t. The equation may therefore be written

$$y = b \sin(k^{1/2}t). \quad . \quad . \quad . \quad . \quad (10)$$

Equations (8) and (10) give the position, and (7) and (9) the velocity, at every instant.

Repulsive force.— If the force is repulsive, the equations of motion must be changed by the substitution of a minus quantity for k. The above solution holds mathematically for this case, but involves imaginaries. This may be avoided by integrating the equations by the method employed in Art. 230. If initial conditions are assumed as in the above solution of the case of attraction, the values of x and y are

$$x = \tfrac{1}{2}a(e^{k^{1/2}t} + e^{-k^{1/2}t})\,; \quad . \quad . \quad . \quad (11)$$

$$y = \tfrac{1}{2}b(e^{k^{1/2}t} - e^{-k^{1/2}t}). \quad . \quad . \quad . \quad (12)$$

In these equations k means the repulsive force per unit mass at unit distance from the center, and $b = V/\sqrt{k}$.

Examples.

1. Show by eliminating t between equations (8) and (10) that the path is an ellipse whose principal diameters lie in the coördinate axes. Determine the values of the principal semi-diameters.

2. Determine the time of describing the whole ellipse. Prove that this depends upon the intensity of the force (*i. e.*, its value at a given distance from the center), and is independent of the initial conditions.

3. What are the dimensions of the constant k?

4. In case of a repulsive force, show that the path is an hyperbola. Determine its equation.

301. Central Force Varying According to Any Law.— Returning to the general case of motion under a force directed toward a fixed point, certain general principles may be deduced which hold without restricting the law of variation of the force.

Using polar coördinates and resolving along and perpendicular to the radius vector, the equations of motion are (Art. 299)

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{P}{m}; \quad . \quad . \quad . \quad (1)$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0. \quad . \quad . \quad . \quad . \quad (2)$$

Here P is the magnitude of the force. To determine the motion these equations must be integrated. For the complete integration, P must be a known function of one or more of the variables r, θ, t. In the most important case P is a function of r only. One important result may, however, be obtained for any case of motion under a central force.

From equation (2), multiplying by r and integrating, we have

$$r^2\frac{d\theta}{dt} = h = \text{constant}. \quad . \quad . \quad . \quad (3)$$

To interpret this result, let A denote the area swept over by the radius vector, in moving from some given position. Thus, in Fig. 134, let N be the position of the particle at time t, $ON = r$, $NOX = \theta$; and let MN be a portion of the path, described in the direction MN. Then

$$A = \text{area } MON.$$

In time Δt let the particle move to N'; the increments of θ and A are

$$\Delta\theta = \text{angle } NON', \quad \Delta A = \text{area } NON'.$$

Approximately,

$$\Delta A = \tfrac{1}{2}r^2\Delta\theta, \quad \text{or} \quad \Delta A/\Delta t = \tfrac{1}{2}r^2(\Delta\theta/\Delta t).$$

In the limit, as Δt is made to approach zero, the equation is exact; that is,

$$r^2(d\theta/dt) = 2(dA/dt).$$

Equation (3) may therefore be written

$$dA/dt = h/2 = \text{constant}. \quad . \quad . \quad . \quad (4)$$

The quantity dA/dt may be called the *areal velocity* of the particle, being the rate at which the radius vector is describing area at any instant. The last equation therefore shows that the areal velocity is constant; and this conclusion evidently holds for all cases of motion of a particle in a plane, so long as the acceleration is directed toward a fixed point.

Integrating equation (4) between limits corresponding to instants t_1 and t_2,

$$A_2 - A_1 = \tfrac{1}{2}h(t_2 - t_1).$$

That is, the area described by the radius vector during any interval is proportional directly to the interval.

The constant h is the *double areal velocity*. Its value is known in any case if the velocity in some one position is known in magnitude and direction. Thus, at a certain instant let the distance from the center of force be a, and let the resolved part of the velocity at right angles to the radius vector be V. The areal velocity at that instant is $aV/2$, and therefore $h = aV$.

As applied to the motion of a planet under the attraction of the sun, equation (4) expresses Kepler's law that the radius vector describes equal areas in equal times. The law is seen to be true, not only when the force follows the law of gravitation, but in every case of central force.

For use in the treatment of certain problems, equation (1) takes a more convenient form if a new variable, $u = 1/r$, be introduced instead of r. This is accomplished as follows:
From equation (3)

$$\frac{d\theta}{dt} = hu^2. \quad . \quad . \quad . \quad . \quad (5)$$

Also, differentiating the equation $r = 1/u$,

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}$$

(putting for $d\theta/dt$ its value hu^2). Again,

$$\frac{d^2r}{dt^2} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}.$$

Substituting the values of $d\theta/dt$ and d^2r/dt^2 in (1), we have finally

$$\frac{P}{m} = h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3. \quad . \quad . \quad . \quad (6)$$

Equations (5) and (6) may be used instead of (1) and (2).

To determine the law of variation of the force when the path is given.—If the polar equation of the path is given, $d^2u/d\theta^2$ may be found by differentiation. By substituting its value in equation (6), the force may be determined as a function of θ, u or r.

EXAMPLES.

1. Show that a particle cannot describe a straight line under a force directed toward or from a fixed point not in the path.

[Applying the above method of determining the law of force, it will appear that $P = 0$.]

2. If the orbit is a circle passing through the center of force, show that the force varies inversely as the fifth power of the distance.

3. A particle describes a conic section under the action of a force which is always directed toward a focus. Show that the force varies inversely as the square of the distance.

302. Force Varying Inversely as the Square of the Distance.— The most important case of motion under a central force is that in which the force varies inversely as the square of the distance from the fixed point. In this case it is most convenient to employ equations (5) and (6) of Art. 301. Let the attraction per unit mass at unit distance from the center be k; then $P/m = k/r^2 = ku^2$, and equation (6) reduces to the form

$$\frac{k}{h^2} = \frac{d^2u}{d\theta^2} + u. \qquad (7)$$

This, and the equation

$$\frac{d\theta}{dt} = hu^2, \qquad (8)$$

are equivalent, in the present case, to the general differential equations of motion.

The path of the particle may be found by the integration of equation (7). To accomplish this, put $u = z + k/h^2$, and there results

$$\frac{d^2z}{d\theta^2} = -z.$$

Proceeding as in Art. 229, the first integration gives

$$\left(\frac{dz}{d\theta}\right)^2 = -z^2 + C^2; \qquad (9)$$

and the second,

$$z = C \sin (\theta + \alpha).$$

Here C and α are constants whose values must be found from the initial conditions. Restoring the value of z,

$$u = \frac{k}{h^2} + C \sin (\theta + \alpha). \qquad (10)$$

The form of this equation shows that the curve is a conic section, the focus being at the pole or center of force. To simplify its form, let C and α be determined from the condition that when $\theta = 0$, r is equal to a, and the velocity is perpendicular to the radius vector. The last condition makes dr/dt, du/dt and $du/d\theta$ all zero when $\theta = 0$. From (10),

$$\left(\frac{du}{d\theta}\right)^2 = C^2 \cos^2 (\theta + \alpha).$$

Hence the condition $du/d\theta = 0$ when $\theta = 0$ gives

$$\cos \alpha = 0; \quad \sin \alpha = 1.$$

Also from (10), applying the condition that $u = 1/a$ when $\theta = 0$,

$$C = \frac{1}{a} - \frac{k}{h^2}.$$

The equation of the path is therefore

$$u = \frac{k}{h^2} + \left(\frac{1}{a} - \frac{k}{h^2}\right) \cos \theta = \frac{k}{h^2}\left[1 + \left(\frac{h^2}{ak} - 1\right) \cos \theta\right].$$

This represents a conic section of eccentricity $h^2/ak - 1$ and semi-parameter h^2/k, the focus being at the pole of the system of coördinates and the principal diameter lying in the initial line.

Since a conic section is an ellipse, a parabola, or an hyperbola, according as the eccentricity is less than, equal to, or greater than unity, it is seen that the orbit is

an ellipse if $h^2/ak < 2$;

a parabola if $h^2/ak = 2$;

an hyperbola if $h^2/ak > 2$.

Since h depends upon the velocity in the initial position, it is seen that if the initial velocity has a certain value the orbit is a parabola; if it is less than this value the orbit is an ellipse; and if greater, the orbit is an hyperbola. To find this critical velocity, let V denote the velocity when $r = a$ and $\theta = 0$, its direction being perpendicular to the radius vector. The double areal velocity h has the value aV; hence $h^2/ak = aV^2/k$, and the above relations become:

$V^2 < 2k/a$ for ellipse;

$V^2 = 2k/a$ for parabola;

$V^2 > 2k/a$ for hyperbola.

Examples.

1. A body moves under the action of a central force which varies inversely as the square of the distance. At a certain instant it is 10 ft. from the center of attraction and moving at right angles to the radius vector at the rate of 5 ft.-per-sec. In this position the force is 10 poundals per pound of mass. Write the equation of the path, and determine whether it is an ellipse, a parabola or an hyperbola.

2. If v is the velocity of the particle when at a distance r from the center of attraction, show that the path is an ellipse, a parabola or an hyperbola, according as v^2 is less than, equal to, or greater than $2k/r$; this result being independent of the direction of v.

3. Prove that $v^2 - 2k/r$ is constant when the force varies inversely as the square of the distance.

4. Integrate equation (6), Art. 301, for the case in which the force varies inversely as the cube of the distance.

§ 5. *Constrained Motion.*

303. Meaning of Constrained Motion.— The motion of a body is said to be *constrained* if certain conditions are imposed upon it, while the forces which must act in order that the motion may conform to those conditions are not specified.

A bead sliding on a wire of any form which is either at rest or moves in a specified manner furnishes an example of constrained motion. Another case is that of a body sliding on an inclined plane which is either at rest or is moving in a given manner.

The laws of motion and the general equations deduced from them, in any of the forms given in the foregoing discussions, may be applied in the solution of problems in constrained motion. The constraint is always produced by the action of forces. The feature which distinguishes constrained motion is that *certain of the forces are not given directly, but are given indirectly by specifying their effect upon the motion.* Such forces are called *constraining forces.*

304. General Method.—In dealing with a case of constrained motion of a particle, we are to apply the general differential equations of motion, the constraining forces being introduced as unknown quantities. In addition there will be one or more equations expressing the conditions which are to be imposed upon the motion. If rectangular coördinates are used, we have (a)

$$m\ddot{x} = X, \qquad (1)$$

$$m\ddot{y} = Y, \qquad (2)$$

in which X and Y involve the unknown constraining forces as well as the known forces; and (b) certain *equations of condition* which may involve any or all of the quantities $x, y, t, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}$.

The only case which will here be considered is that in which the path is assigned. The equation of condition is then of the form

$$f(x, y) = 0. \qquad (3)$$

Before discussing this problem in its general form, the simple case of motion in a straight line will be considered.

305. Motion on a Smooth Inclined Plane Under Gravity.— Let a body of mass m pounds be placed upon a smooth plane surface, inclined to the horizontal at an angle α. The forces acting upon it are its weight and the pressure exerted by the body which it touches. The former is mg poundals vertically downward, and the latter is perpendicular to the plane but of unknown magnitude (say N poundals). The force N is called into action only to resist a tendency of the body to pass through the surface; it cannot cause the body to leave the surface, for it ceases to act as soon as the bodies separate. The motion is thus "constrained" to follow the plane. It will be assumed also that the motion is confined to a vertical plane, which is taken as the plane of the figure.

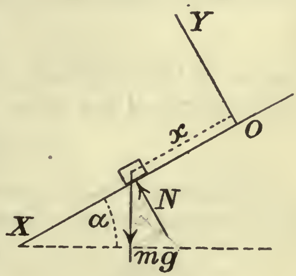


FIG. 135.

Referring to rectangular coördinates, let the axis of x (OX, Fig. 135) lie in the plane, the positive direction being downward along the plane. The equations of motion are

$$m\ddot{x} = mg \sin \alpha; \qquad (1)$$

$$m\ddot{y} = N - mg \cos \alpha. \qquad (2)$$

But since the path is the straight line $y = 0$, so that $\ddot{y} = 0$, equation (2) becomes

$$N - mg \cos \alpha = 0. \qquad (3)$$

Equation (3) serves to determine N, while (1) determines the motion.

Comparing equation (1) with the equation of motion for a body falling vertically, it is seen that the results deduced in Art. 227 may be applied to the present case by substituting $g \sin a$ for g.

The three cases in which the initial velocity is zero, down the plane, and up the plane, may be discussed as were the corresponding cases in Art. 227.

306. Motion on a Rough Plane Under Gravity.— If the plane is rough, friction must be included among the forces entering the equations of motion. While the body is sliding, the force of friction is equal to μN, μ being the coefficient of friction (Art. 129); the direction of this force is opposite to the sliding. Taking axes as in Fig. 136, the equations of motion are

$$m\ddot{x} = mg \sin a - \mu N; \quad . \quad . \quad . \quad (1)$$

$$m\ddot{y} = N - mg \cos a. \quad . \quad . \quad . \quad (2)$$

The latter equation becomes

$$N - mg \cos a = 0, \quad . \quad . \quad . \quad . \quad (3)$$

because of the condition $y = 0$. Equation (3) gives the value of N; substituting this in (1),

$$\ddot{x} = g(\sin a - \mu \cos a). \quad . \quad . \quad . \quad (4)$$

The integration of equation (4) gives results agreeing with those of Art. 227, with the substitution of $g(\sin a - \mu \cos a)$ for g. This equation applies only to the case in which the motion is down the plane. If it is up the plane the direction of the force μN must be reversed. If x is still taken as positive downward along the plane, the case of motion up the plane gives the equation

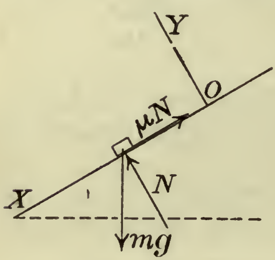


FIG. 136.

$$\ddot{x} = g(\sin a + \mu \cos a). \quad . \quad (5)$$

It should be noticed that, in the case of motion down the plane, the velocity will increase or decrease, according as the second member of (4) is positive or negative. Between these cases is that in which

$$\sin a - \mu \cos a = 0, \quad \text{or} \quad \mu = \tan a,$$

which reduces equation (4) to the form

$$\ddot{x} = 0.$$

This is the case of equilibrium; the particle will remain at rest or will move uniformly down the plane.

Uniform motion *up* the plane is impossible.

EXAMPLES.

1. A body slides down a smooth plane whose inclination to the horizon is 20°. How far does it move during the first 2 sec. after starting from rest? *Ans.* 22 ft.

2. A body slides down a smooth plane whose inclination to the horizon is 25°. What is the velocity when it reaches a position 12 ft. lower than its position of rest? *Ans.* 27.8 ft.-per-sec.

3. Solve Ex. 2 if the inclination of the plane is 40°, the data being otherwise unchanged.

4. A body starting from rest slides down a smooth plane whose inclination to the horizon is α. What is its velocity after moving a distance whose vertical projection is h? Show that the velocity acquired in falling to a given level is independent of the inclination of the plane.

5. A body is projected up a smooth plane inclined 30° to the horizon with a velocity of 200 c.m.-per-sec. What is its velocity after moving 40 c.m.? When will it come to rest, and when will it return to the point from which it is projected?

Ans. 28 c.m.-per-sec.; after 0.41 sec.; after 0.82 sec.

6. A body is projected with a velocity V up a smooth plane whose inclination is α. When will it come to rest, when will it return to the starting point, and how high will it rise? Prove that it will rise to the same height as if projected vertically upward with the same speed.

7. Solve Ex. 1 assuming the plane rough, the angle of friction being 10°. *Ans.* 11.4 ft.

8. A body is projected with a velocity of 20 ft.-per-sec. down a plane whose inclination is 25°, the coefficient of friction being 0.4. Determine the position and velocity after 2 sec.

Ans. 43.9 ft. from starting point; 23.9 ft.-per-sec.

9. In Ex. 8, if $\mu = 0.5$, when will the body come to rest?

Ans. After 20.4 sec.

10. In Ex. 8, if the angle of friction is 25°, determine the motion.

11. In Ex. 8, let the direction of the initial velocity be reversed, and let the angle of friction be 25°. Determine the motion.

Ans. The body will come to rest after 0.73 sec. and will remain at rest.

12. A body is placed at rest on a rough plane, and the inclination of the plane is increased until sliding begins. If the coefficient of friction is 0.2, what is the angle of incipient sliding?

13. A body is projected up an inclined plane. What condition must be satisfied in order that it may slide down after coming to rest?

14. A body is projected up a plane whose inclination to the horizon is 30°. The angle of friction is 20°. If the initial velocity is 20 ft.-per-sec., (*a*) when will the body come to rest, (*b*) when will it return to the initial position, and (*c*) with what velocity will it pass through the initial position?

Ans. (*a*) At end of 0.76 sec. (*b*) 1.60 sec. after coming to rest. (*c*) 9.52 ft.-per-sec.

15. A body is projected on a horizontal plane with a velocity of 50 ft.-per-sec. It comes to rest after 6 sec. What is the degree of roughness of the plane?

16. A stone thrown horizontally on a sheet of ice with a velocity of 120 ft.-per-sec. comes to rest after sliding 1,400 ft. Assuming the coefficient of friction to be independent of the velocity, determine its value. *Ans.* 0.16.

17. Write the equations of motion of a body sliding on a smooth inclined plane, under the action of no force except gravity and the pressure of the plane, assuming that the motion is not confined to a vertical plane. Show that the path is a parabola.

18. In the case described in Ex. 17, show that the motion is given by equations like those deduced in Art. 295 for a projectile, with the substitution of $g \sin \alpha$ for g, α being the inclination of the plane to the horizon.

307. Motion of a Bead on a Smooth Wire.—The general case in which a particle is constrained to move in a given plane curve will now be considered. To fix the ideas, let the particle be a bead sliding on a smooth wire bent into any plane curve, and suppose all forces acting on the bead to be known, *except the pressure exerted by the wire.* This pressure is a "passive resistance" (Art. 41); it comes into action to resist any tendency of the bead to leave the wire. The direction of the pressure exerted by the wire upon the bead is normal to the path of the particle.

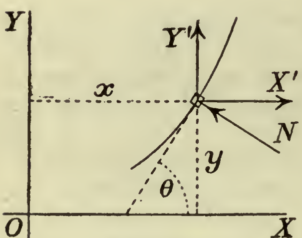


FIG. 137.

Let N represent the magnitude of this unknown normal pressure, regarded as positive if directed toward the concave side of the path. (See Fig. 137.) Let Q denote the resultant of all forces applied to the bead except the normal pressure N.

Two independent equations of motion are now to be written, ob-

tained by resolving resultant force and resultant acceleration in any two directions. Two sets of equations will be given.

(1) *Resolution along fixed rectangular axes.*—Choosing any pair of rectangular axes, let θ denote the angle between the tangent to the path and the axis of x. Then the normal pressure N makes with the x-axis the angle $90° + \theta$, and with the y-axis the angle θ (Fig. 137). The axial components of N are therefore

$$-N \sin \theta \quad \text{and} \quad N \cos \theta.$$

But if s means the length of the path, measured from some fixed point, $\sin \theta = dy/ds$, $\cos \theta = dx/ds$; therefore

$$-N(dy/ds) = x\text{-component of } N;$$

$$N(dx/ds) = y\text{-component of } N.$$

Let X', Y' denote the axial components of the known force Q, and X, Y the axial components of the resultant of *all* forces acting on the bead (*i. e.*, the resultant of Q and N). Then

$$X = X' - N(dy/ds);$$

$$Y = Y' + N(dx/ds).$$

The equations of motion then become (Art. 287)

$$m(d^2x/dt^2) = X' - N(dy/ds); \qquad . \quad . \quad (1)$$

$$m(d^2y/dt^2) = Y' + N(dx/ds). \qquad . \quad . \quad (2)$$

These two equations, together with the equation of the path, serve to determine the motion.

(2) *Resolution along tangent and normal.*—Let Q_t and Q_n represent the tangential and normal components of Q; then if forces and acceleration be resolved along the tangent and normal to the curve at each instant (Art. 289), the equations of motion become

$$m(dv/dt) = Q_t; \qquad . \quad . \quad . \quad . \quad (3)$$

$$m(v^2/R) = N + Q_n. \qquad . \quad . \quad . \quad (4)$$

In equation (4), R denotes the radius of curvature of the path at the position of the particle.

Equations (3) and (4) are just equivalent to (1) and (2), and either set may be used, as most convenient.

308. Bead Sliding on Smooth Wire Under Gravity.—Let the resultant applied force Q be constant in magnitude and direction. If

the x-axis is taken in the direction of this force, $X' = Q$, $Y' = 0$. If the force Q is the weight of the particle, $Q = mg$, and the equations become

$$m(d^2x/dt^2) = mg - N(dy/ds); \quad . \quad . \quad (5)$$

$$m(d^2y/dt^2) = N(dx/ds). \quad . \quad . \quad . \quad . \quad (6)$$

An important result may be deduced from these equations, irrespective of the form of the curve.

Multiplying (5) by dx and (6) by dy and adding, there results

$$m\left(\frac{dx}{dt}\frac{d^2x}{dt^2}dt + \frac{dy}{dt}\frac{d^2y}{dt^2}dt\right) = mg\,dx,$$

the integration of which gives

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\left(\frac{dy}{dt}\right)^2 = gx + \text{constant},$$

or
$$v^2 = 2gx + \text{constant},$$

if v is the speed. Taking the integral between limits x_1 and x_2 for x, and limits v_1 and v_2 for v,

$$v_2^2 - v_1^2 = 2g(x_2 - x_1) = 2gh,$$

if h is the height through which the body falls while the velocity changes from v_1 to v_2.

Comparing this result with that of Ex. 1, Art. 227, it is seen that the change in the speed of the particle is the same as if it had fallen freely through the same vertical distance under the action of gravity.

The further consideration of this problem will be restricted to the case in which the path of the particle is a circle. The problem then becomes identical with that of the motion of a simple pendulum.

309. Simple Pendulum.— A particle suspended from a fixed point by means of a weightless rigid rod or flexible inextensible string, and acted upon by no forces except gravity and the force exerted by the rod or string, is called a simple pendulum.

Such a pedulum exists only ideally. An actual pendulum consists of a body suspended by a bar or string possessing weight. The motion of any actual pendulum, if frictional resistances be neglected, is the same as that of some simple pendulum, so that the discussion of the ideal simple pendulum is of practical value as well as theoretical interest.

Consider then the case of a particle suspended by a perfectly

flexible but inextensible string. So long as the string remains tense the path must be a circle with the point of suspension as center; and the force exerted by the string upon the particle is directed toward the center.

Let O (Fig. 138) be the point of suspension, A the lowest position occupied by the particle during its motion, and B the position at any instant. Represent the angle AOB by θ, and let $OA = l$. Also let m denote the mass of the particle, and N the magnitude of the force exerted by the string, its direction being toward the center. The only other force acting upon the particle is its weight, a force of magnitude mg, directed downward. Let s denote the length of the arc AB; then $s = l\theta$. (Both θ and s will be negative if the particle passes to the left of A.)

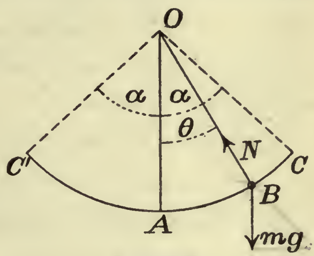


FIG. 138.

Equations (5) and (6) of Art. 308 are now applicable; but it will be better to resolve along the tangent and normal to the path. The equations thus take the forms (3) and (4) (Art. 307). Resolving in the direction of the tangent,

$$m(d^2s/dt^2) = -mg \sin \theta; \quad . \quad . \quad . \quad (7)$$

and resolving in the direction BO,

$$m(v^2/l) = N - mg \cos \theta. \quad . \quad . \quad . \quad (8)$$

Since $s = l\theta$, (7) may be written

$$d^2\theta/dt^2 = -(g/l) \sin \theta. \quad . \quad . \quad . \quad (9)$$

The integration of this equation determines the motion, while from (8) N may be determined when the motion is known. This would constitute the complete solution of the problem. The complete integration of (9) will not be shown, since it involves an elliptic integral. One integration, however, may readily be performed, and the value of N determined. If the range of the motion is small in comparison with the radius (*i. e.*, if the angle θ is small), a complete approximate solution may be made which is practically correct.

(1) *Exact partial solution.*—Mutiplying both members of equation (9) by $d\theta$ and integrating, there results

$$\tfrac{1}{2}(d\theta/dt)^2 = (g/l) \cos \theta + C.$$

To determine C, let α be the value of θ when the particle is at rest in its highest position; then $C = -(g/l)\cos\alpha$, and the equation becomes

$$(d\theta/dt)^2 = 2(g/l)(\cos\theta - \cos\alpha). \quad . \quad . \quad (10)$$

The velocity in any position is therefore

$$v = l(d\theta/dt) = \sqrt{2gl(\cos\theta - \cos\alpha)}. \quad . \quad (11)$$

The value of N may now be determined by substituting the value of v^2 in (8). That is,

$$N = m(v^2/l) + mg\cos\theta = m[2g(\cos\theta - \cos\alpha) + g\cos\theta],$$

or

$$N = mg(3\cos\theta - 2\cos\alpha). \quad . \quad . \quad (12)$$

(2) *Approximate complete solution.*—Suppose the angle θ to remain very small throughout the motion; then, approximately,

$$\sin\theta = s/l,$$

and equation (7) becomes

$$d^2s/dt^2 = -(g/l)s. \quad . \quad . \quad . \quad . \quad (13)$$

Multiplying by $2(ds/dt)dt$ and integrating,

$$(ds/dt)^2 = (g/l)(a^2 - s^2). \quad . \quad . \quad . \quad (14)$$

Here the constant of integration has been determined on the assumption that $s = a$ when the particle is at rest in its highest position. Equation (14) may be written

$$dt\sqrt{g/l} = ds/\sqrt{a^2 - s^2}.$$

Integrating, and determining the constant by the condition that $s = 0$ when $t = 0$,

$$t\sqrt{g/l} = \sin^{-1}(s/a) - \sin^{-1}(0).$$

Taking sine of each member and reducing,*

$$s = a\sin(t\sqrt{g/l}). \quad . \quad . \quad . \quad (15)$$

This result is identical in form with that found in Art. 229 for the motion of a particle in a straight line under a central attractive force varying directly as the distance. The motion of a simple pendulum when the range is small, is thus a *harmonic oscillation.*

---

* It will be noticed that the conditions assumed in determining constants of integration leave it uncertain whether the plus or the minus sign should be prefixed to the second member of (15). If it be further assumed that s is plus for small positive values of t (*i. e.*, that the particle is moving in the positive direction through A when $t = 0$), the plus sign must be used. See Art. 229.

Time of vibration and of oscillation.— If C and C' (Fig. 139) are the extreme positions of the particle, the passage from C to C' or from C' to C is called a *vibration*, while the passage from C to C' and back to C is called an *oscillation*. The particle is at C as often as $s = a$, and at C' as often as $s = -a$. While s changes from a to $-a$, $\sin(t\sqrt{g/l})$ changes from $+1$ to -1, and $t\sqrt{g/l}$ changes by 180° or π. Hence the time of one vibration is

$$T = \pi\sqrt{l/g}, \quad . \quad . \quad . \quad (16)$$

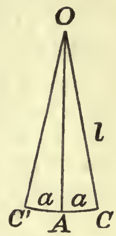


FIG. 139.

while the time of a complete oscillation is $2T$.

Seconds pendulum.— A seconds pendulum is one that vibrates once every second. Its length may be found from (16) if g is known; thus, if $T = 1$, $l = g/\pi^2$.

Determination of g.— By determining the time of vibration of a pendulum of known length, the value of g may be determined from equation (16).

EXAMPLES.

1. What is the length of the seconds pendulum at a place where the value of g is 32.2 ft.-per-sec.-per.-sec.?

2. If g_1 and g_2 are the values of g at the surface of the earth and at an elevation of h ft. above the surface respectively, what is the relation between g_1 and g_2?

Ans. If P_1 and P_2 are the values of the earth's attraction upon the body in the two positions, the law of gravitation (Art. 176) gives the proportion

$$P_1/P_2 = (R + h)^2/R^2.$$

Or, since $P_1/P_2 = g_1/g_2$,

$$g_1/g_2 = (R + h)^2/R^2.$$

3. If h is small compared with R, show that the result of Ex. 2 reduces approximately to the form $g_2 = g_1(1 - 2h/R)$.

4. If T_1 and T_2 are the times of vibration of the same pendulum at two places of which the second is h ft. higher than the first, what is the relation between T_1 and T_2? Show that this relation is approximately expressed by the equation $T_1 = T_2(1 - h/R)$.

5. Show how the last result may be used in determining the difference of elevation between two points in the same neighborhood, the same pendulum being swung in the two places.

6. If N_1 is the number of vibrations of a pendulum in any time at a certain place, and N_2 the number of vibrations of the same pendulum in an equal time at a place h ft. higher, show that, approximately, $h/R = (N_1 - N_2)/N_1$.

7. At sea-level a pendulum beats seconds. At the top of a mountain it beats 86,360 times in 24 hours. What is the height of the mountain?

8. Determine roughly a value of g by swinging a weight suspended by a string.

9. A body whose mass is 1 lb. is suspended from a fixed point by a string 12 ft. long. The string is swung to a position 60° from the vertical and the body released. Determine the velocity when the body is in its lowest position; also when 2 ft. above its lowest position.

10. In Ex. 9, determine the tension in the string in each of the positions specified; also in the initial position.

Ans. Tension in lowest position is twice the weight of the body; in highest position, half the weight of the body.

11. Two bodies A and B are connected by a flexible, inextensible string which passes over a smooth pulley C. The body A is vertically below C and rests on a horizontal plane. The body B is held on a level with C, the string being tight, and is then allowed to fall under the action of gravity. How far will it fall before A leaves the floor?

Ans. Let m, m' be the masses of A and B respectively, and let $\theta = ACB$. Then at the instant when A is lifted, $\cos\theta = m/3m'$. If $m > 3m'$, the body A will not be lifted.

310. Uniform Circular Motion.— If a particle describes a circle of radius r with uniform speed v, its acceleration is at every instant directed toward the center of the circle, and is equal to v^2/r. Therefore, from the general equation of motion, the resultant of all forces acting upon the particle is a force mv^2/r directed toward the center.

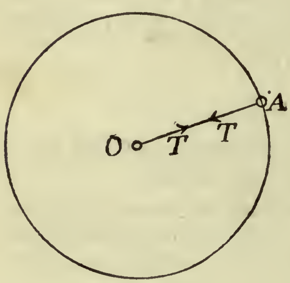


FIG. 140.

Let A (Fig. 140) be the particle, and suppose it to be attached to an inextensible string AO, of which the end O is fastened to a fixed peg. If the particle be projected in a direction at right angles to AO with a velocity v, and if no force acts upon it except that due to the string, the equations of motion obtained by resolving along the tangent and normal to the path (Art. 289) are

$$m(dv/dt) = 0, \quad m(v^2/r) = T;$$

T being the force exerted upon the particle by the string. The first equation shows that the speed v remains constant; the second gives the value of T. The string thus exerts upon the particle a force whose direction is always toward the point O and whose magnitude is mv^2/r. By the law of action and reaction, the particle exerts upon the string a force mv^2/r in the direction OA. The string sustains a tension whose value is uniform throughout its length; *i. e.*, if it be conceived to be divided by a section at any point, the two portions exert upon each other forces which are equal and opposite (Art. 43), each being equal to mv^2/r. The peg is pulled by the string in the direction OA with a force* equal to mv^2/r.

The value of the resultant force acting upon the particle may be expressed in terms of its angular velocity (Art. 281) about the center of the circle. If the angular velocity is ω, $v = r\omega$, and

$$mv^2/r = mr\omega^2.$$

Examples.

1. A body of 2 lbs. mass is swung in a circle of 2 ft. radius by a string which is capable of sustaining 2 lbs. against gravity. At what rate may it move without breaking the string?

Ans. About 8 ft.-per-sec.

2. A locomotive of 250,000 lbs. mass describes a curve of 2,000 ft. radius at the rate of 30 miles-per-hour. What is the resultant force acting upon it? What horizontal pressure does it exert upon the rail? *Ans.* Resultant force = 7,520 lbs. toward the center.

3. A locomotive of mass m describes a curve of radius r with speed v. (*a*) Determine the resultant force acting on the locomotive. (*b*) Determine the magnitude and direction of the resultant pressure between the track and the locomotive. (*c*) What should be the difference in elevation of the rails in order that there shall be no tendency of the locomotive to slide laterally?

Ans. (*a*) The resultant force acting upon the locomotive is a force mv^2/r directed toward the center of the curve. This resultant is made up of two components,—a downward force mg (the weight of the body) and the supporting force exerted by the track. (*b*) This supporting force assumes such magnitude and direction that, when combined with the weight, it gives the resultant mv^2/r toward the center. Hence its magnitude is

---

*This force acting upon the peg is sometimes called the "centrifugal" force. It seems better, however, to reserve this term for another use, in connection with problems in relative motion.

$$\sqrt{(mv^2/r)^2 + (mg)^2} = m\sqrt{(v^2/r)^2 + g^2};$$

and its inclination to the vertical is $\tan^{-1}(v^2/rg)$. (*c*) The pressure between the rails and the locomotive should be normal to the surface determined by the tops of the rails; hence this surface should make with the horizontal the angle whose tangent is v^2/rg. If the horizontal distance between centers of rails is a, the outer rail should be higher than the inner by v^2a/gr.

4. A horizontal platform rotates about a vertical axis at the uniform rate of 6 revolutions per min. A body of 20 lbs. mass rests upon the platform at a point 16 ft. from the axis of rotation. (*a*) What is the resultant force acting upon the body? (*b*) Of what actual forces is this resultant composed? (*c*) How great must be the coefficient of friction to prevent the body from sliding?

Ans. (*a*) 126.3 poundals toward the axis of rotation. (*b*) The actual forces are the weight ($20g$ poundals) and the pressure of the supporting platform. The normal component of this pressure is $20g$ poundals upward, the tangential component is 126.3 poundals toward the axis of rotation. (*c*) The coefficient of friction must be at least $126.3/20g = 0.196$.

5. In Ex. 4, if the coefficient of friction is 0.5, how far from the axis may the body be placed without sliding? *Ans.* 40.8 ft.

6. In Ex. 4, if the body rests upon a smooth surface which is fixed to the platform, what must be the slope of the surface?

7. In Ex. 4, if the body is suspended by a string from a support fixed to the platform, what direction will the string assume, and what tension will it sustain?

Ans. The lower end of the string will swing directly away from the axis of rotation until the string is inclined 11° 6′ to the vertical. The tension will be 656 poundals.

8. A platform rotates about a vertical axis with uniform angular velocity ω. A body distant x from the axis is placed on a smooth surface whose inclination is such that the body rests without sliding. Show that the surface is inclined to the horizontal at an angle whose tangent is $x\omega^2/g$. The surface slopes directly toward the axis.

9. Using the result of Ex. 8, show that a smooth surface upon which a body would rest wherever placed must be a surface generated by a parabola which rotates with the platform and whose principal diameter lies in the axis of rotation.

If x, y are the coördinates of a section of the surface by a plane containing the axis of rotation, the origin being at the point in which the axis pierces the surface, the result of Ex. 8 gives

$$dy/dx = (\omega^2/g)x.$$

Integrating, $$y = (\omega^2/2g)x^2.$$

10. Two bodies, each of 5 lbs. mass, are connected by an elastic string which passes through a straight tube 3 ft. long and by its

tension holds the bodies against the ends of the tube. The system rotates uniformly about a vertical axis perpendicular to the axis of the tube at its middle point. The natural length of the string is such that a pull equal to the weight of 2 lbs. is required to stretch it to the length of the tube. What is the pressure of each body against the tube when the system is making 10 revolutions per minute? (Take $g = 32.2$ ft.-per-sec.-per-sec.)

Ans. 56.2 poundals or 1.75 pounds-force.

11. Let the system described in Ex. 10 rotate about a vertical axis 6 ins. from the middle point of the tube. Determine the pressure of each body against the tube.

Ans. The body nearer the axis presses against the tube with a force of 58.9 poundals; the other with a force of 53.4 poundals.

12. The system and axis of rotation being as described in Ex. 10, suppose the angular velocity gradually to increase. When will the bodies cease to press against the tube?

Ans. When the angular velocity reaches 2.93 rad.-per-sec.

13. If the axis of rotation is located as in Ex. 11, and the angular velocity gradually increases, which body will leave the tube first? When this occurs, what will be the pressure of the other body on the tube?

Ans. When the angular velocity reaches 2.54 rad.-per-sec. the body 2 ft. from the axis of rotation will cease to press against the tube. The pressure of the other against the tube will then be equal to half the tension in the string or 32.2 poundals.

14. A particle suspended from a fixed point by an inextensible string and acted upon by no force except its weight and the pull of the string describes a horizontal circle. If a is the length of the string, θ the angle it makes with the vertical, ω the angular velocity, T the tension in the string, and m the mass of the particle, show that $\cos\theta = g/a\omega^2$, $T = ma\omega^2$.

15. In Ex. 14 let the mass of the particle be 1 kilogr. and the length of the string 1 met. If the number of revolutions per minute is 40, determine θ and T. (Assume $g = 981$ C. G. S. units.)

Ans. $\theta = 56°$ (nearly); $T = 1.755 \times 10^6$ dynes $= 1.788$ kilograms-weight.

311. Effect of the Earth's Rotation Upon Apparent Weights of Bodies.— If by the weight of a body is meant the gravitational pull of the earth upon it, the apparent weight differs from the true weight because of the earth's rotation.

Let Fig. 141 represent a meridian of the earth, NS being the polar diameter and B a point in the equator. The form of the meridian is known to be nearly elliptical, the length of the polar radius being 6.356×10^8 c.m. and that of the equatorial radius

6.378×10^8 c.m., very nearly. In Fig. 141 the ellipticity is greatly exaggerated. If the earth were spherical and of uniform density, or of density varying only with the distance from the center, the attraction upon a body at the surface would be directed toward the center (Art. 183). In fact the direction differs from this. In Fig. 141, let AH' be the direction of the earth's attraction upon a body at A, and let P' denote the magnitude of the attraction upon a body of mass m.

To fix the ideas, suppose the body to be suspended by a string AC attached to a spring balance, and let P be the supporting force as measured by the balance. If the body were in equilibrium, P

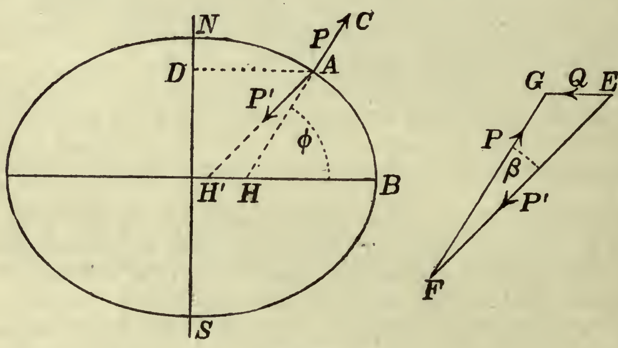


FIG. 141.

would be equal and opposite to P'. But since the body moves with the earth in its daily rotation, the string assumes such a direction and the supporting tension such a magnitude that the resultant of P and P' is just sufficient to maintain the actual motion of the body.

Let r = radius of circle described by body, ω = angular velocity of earth's rotation; then the resultant force acting upon the body has the direction AD and the magnitude $Q = mr\omega^2$. The value of Q may thus be computed, since ω is a known constant and r is known for any latitude.

The relation between P, P' and Q is represented by the vector triangle EFG, in which EF is parallel to AH', FG to AC, and EG to AD. The value of P is determined by experiment; Q may be computed as above; and P' may then be determined by solving the triangle EFG.

The force P is equal and opposite to the "apparent" weight of

the body. Its direction is normal to the earth's surface at A (*i. e.*, to a level surface, such as the surface of still water). The angle ϕ between AC and the plane of the equator is the latitude of the place. It will be shown presently (examples 3 and 4 below) that Q is very small in comparison with P, so that the angle EFG is also very small; assuming this to be true, the value of the angle and of the difference between P and P' may be determined to a close approximation in the following simple form:

$$\text{angle } EFG = \beta = (Q \sin \phi)/P; \qquad (1)$$

$$P' - P = Q \cos \phi. \qquad (2)$$

Here β is expressed in radians.

Examples.

1. The time of one revolution of the earth is 86,164 mean solar sec. Determine the value of ω, the angular velocity, in rad.-per-sec.

Ans. $\log \omega = 5.8628 - 10$; $\omega = 0.00007292$.

2. Determine the linear velocity of a body at sea level at the equator, using the value of the equatorial radius given above.

Ans. 4.65×10^4 c.m.-per-sec.

3. Determine the difference between the true and the apparent weight of a body of m gr. at the equator.

Ans. For a body at the equator P, P' and Q are all parallel, and $P' - P = Q$. Also, $Q = mv^2/r$, in which $v = 4.65 \times 10^4$ (Ex. 2), $r = 6.378 \times 10^8$. Therefore $Q = 3.391m$ dynes.

4. If the value of g at sea-level at the equator is 978.1 c.m.-per-sec.-per-sec., what would be its value if not modified by the earth's rotation?

The value of g as experimentally determined is the "apparent" acceleration,—*i. e.*, the acceleration on the assumption that the earth is at rest. If P is the apparent weight of a body of mass m,

$$g = P/m.$$

The acceleration which would be due to the true attraction is P'/m. At the equator,

$$P'/m = P/m + Q/m = 978.1 + 3.4 = 981.5.$$

5. Show that if the earth should rotate seventeen times as rapidly as at present, its form and density being unchanged, the apparent weights of bodies at the equator would be reduced nearly to zero.

6. In latitude 40° the radius of a parallel of latitude is very nearly 4.894×10^8 c.m. The value of g at sea-level is about 980.2 C. G. S. units. Determine the influence of the earth's rotation on the magnitude and direction of g.

Using C. G. S. units, the value of Q is $mr\omega^2$, in which $r = 4.894 \times 10^8$, $\omega = 7.292 \times 10^{-5}$ (Ex. 1); that is, $Q = 2.602m$ dynes. From equation (2),

$$P' - P = Q \cos \phi = 2.602m \cos 40^\circ = 1.993m.$$

From the given data, $P = 980.2m$, hence $P' = 982.2m$; and equation (1) gives

$$\beta = (Q \sin \phi)/P' = (2.602 \sin 40^\circ)/982.2$$
$$= 0.001703 \text{ radians} = 5' \, 51''.$$

The rotation of the earth thus changes the magnitude of g by 1.993 C. G. S. units, and its direction by $5' \, 51''$.

7. If the length of a degree of longitude is 78,849 meters in latitude 45°, and the value of g 980.6 C. G. S. units, compute the effect of the earth's rotation on the value of g.

8. If the diminution of the apparent weight of a given body in latitude ϕ is w, and if the value of w at the equator is w', show that $w = w' \cos^2 \phi$ (very nearly).

CHAPTER XVI.

MOMENTUM AND IMPULSE.

§ 1. *Rectilinear Motion.*

312. **Momentum.**— The *momentum* of a particle is a quantity proportional directly to its mass and to its velocity.

Momentum is sometimes called mass-velocity. It is often regarded as a measure of the "quantity of motion" of the particle.

The numerical value of the momentum of a particle of given mass moving with given velocity depends upon the unit in terms of which it is expressed.

Unit of momentum.— For most purposes it is convenient to choose as the unit the momentum possessed by a body of unit mass having the unit velocity. The unit thus defined depends upon the units of mass, length and time. If these are the pound, the foot and the second, respectively, the unit of momentum is that of a body of one pound mass having a velocity of one foot-per-second.

Other units would be the momentum of a particle of one gram mass having a velocity of one centimeter-per-second; and that of a kilogram mass having a velocity of one meter-per-second.

In the discussion which immediately follows, it is to be understood that the motion is restricted to a straight line.

313. **Increment of Momentum.**— If the velocity of a particle varies, so also does the momentum. If the mass is m, and if the velocity has values v_1 and v_2 at the beginning and end of a given interval respectively, the *increment of the momentum* during the interval is

$$mv_2 - mv_1 = m(v_2 - v_1).$$

314. **Acceleration of Momentum.**— The rate of increase of the momentum (or increment of momentum per unit time) is called the *acceleration of momentum.*

If the velocity varies at a uniform rate, receiving an increment Δv in an interval of time Δt, the acceleration of momentum is equal to $m(\Delta v/\Delta t)$. If the velocity changes at a variable rate, the instantaneous value of the acceleration of momentum is $m(dv/dt)$. In either case its value is mp, p being the acceleration. For this reason it is often called *mass-acceleration.*

The most convenient unit of mass-acceleration is that of a unit mass having unit acceleration. With the usual British units of mass, length and time, this unit would be the mass-acceleration of a pound-mass having an acceleration of one foot-per-second-per-second. If the gram and centimeter are taken as units of mass and length respectively, the unit mass-acceleration is that possessed by a particle of one gram mass having an acceleration of one centimeter-per-second-per-second.

315. Impulse of a Constant Force.— The *impulse* of a force which remains constant in direction and magnitude is a quantity proportional directly to the force and to the time during which it acts. The numerical value of the impulse of a given force acting for a given time depends upon the unit in terms of which it is expressed.

Unit impulse.— Let the unit impulse be that of a unit force acting for a unit time. Then the value of any impulse is equal to the product of the force into the time.

This unit depends upon the units of force and of time. In the F. P. S. system the unit impulse is that of a poundal force acting for a second. In the C. G. S. system it is the impulse of a dyne acting for a second. These may be designated as a *poundal-second* and a *dyne-second* respectively. If the engineers' system (Art. 218) is employed, the unit impulse is the *pound-second.*

Impulse is a localized vector quantity (Art. 26), its direction and line of action coinciding with those of the force. Restricting the discussion to forces acting upon a particle moving in a straight line which is the line cf action of all the forces, the only directions to be distinguished are the two opposite directions along the path ; these may be distinguished by signs + and —.

316. Impulse of a Variable Force.— If the magnitude of a force varies continuously throughout a certain interval, its impulse must be computed by integration.

Let the value of the force at any instant t be denoted by P, and let it be required to determine the value of the impulse during the interval from $t = t'$ to $t = t''$. Let this interval be divided into small intervals each equal to Δt; and let $P_1, P_2, \ldots$ denote the values of P at the beginnings of the successive small intervals. Represent the required impulse by P'; then, approximately,

$$P' = P_1 \Delta t + P_2 \Delta t + \ldots \qquad (1)$$

The approximation is closer the smaller Δt is taken. If Δt is made to approach zero, the number of terms in the series (1) increases indefinitely while the value of every term approaches zero. The sum in general approaches a definite value which is the exact value of the impulse P'. That is,

$$P' = \text{limit}\,[P_1\,\Delta t + P_2\,\Delta t + \quad . \quad . \quad . \quad] = \int_{t'}^{t''} P\,dt. \quad . \quad (2)$$

When P is constant this reduces to

$$P' = P(t'' - t'),$$

agreeing with Art. 315.

317. Relation Between Impulse and Momentum.— If m is the mass of a particle acted upon by a single force P directed along the line of motion, the general equation of motion is

$$P = m(dv/dt).$$

Multiplying through by dt and integrating between limits t' and t'',

$$\int_{t'}^{t''} P\,dt = m(v'' - v'), \quad . \quad . \quad . \quad (3)$$

v' and v'' being the values of the velocity at the instants t' and t'' respectively.

The first member of equation (3) is the value of the impulse of the force P during the interval from t' to t''. In the second member, mv' is the value of the momentum at the instant t', and mv'' its value at the instant t''; $m(v'' - v')$ is the increment of the momentum for the interval from t' to t''. Equation (3) therefore expresses the proposition that

The impulse of a force acting alone upon any particle during a given interval is equal to the change of momentum of the particle during that interval.

If several forces act upon the particle, its change of momentum during any interval is equal to the impulse of the resultant force; and this is evidently equal to the algebraic sum of the impulses of the several forces.

EXAMPLES.

1. A body whose mass is m pounds falls vertically under the action of gravity. (*a*) Compute the impulse of the force during t sec. (*b*) What increment of momentum is received during t sec.?

Ans. (*a*) mgt poundal-seconds. (*b*) mgt momentum-units.

2. In Ex. 1, if the body has initially an upward velocity v_1, what is its final velocity? *Ans.* $gt - v_1$ ft.-per-sec. downward.

3. A body of 12 lbs. mass is projected downward with a velocity of 20 ft.-per-sec. Write the equation of impulse and momentum for an interval of 3 sec., and thus determine the velocity at the end of the interval.

4. A particle of mass m is acted upon by a force $P = kt$, t being the time reckoned from a definite instant. Compute the impulse during the interval from $t = 0$ to $t = t'$. Write the equation of impulse and momentum, assuming the velocity to be zero when $t = 0$. Explain the meaning of k.

5. In Ex. 4, let $m = 500$ gr., and let $P = 10{,}000$ dynes when $t = 1$ sec. Write the equation of impulse and momentum and determine the velocity when $t = 8$ sec.

Ans. $v'' = 640$ c.m.-per-sec.

318. Dimensions of Impulse and Momentum.— If units of force, mass, length and time are all chosen arbitrarily, the unit impulse is of dimensions **FT**, and the unit momentum of dimensions **ML/T.** But with such an arbitrary choice of units the equation of impulse and momentum must contain a constant, taking the form

$$(\text{impulse}) = (\text{constant}) \times (\text{change of momentum}),$$

the value of the constant being determined in accordance with the particular fundamental units chosen.

If, as in Art. 317, a kinetic system of units is employed, the unit force has dimensions* **ML/T²**, and the unit impulse is therefore of dimensions **ML/T.** The equation

$$\text{impulse} = \text{change of momentum}$$

thus becomes homogeneous, both members being of the same dimensions in terms of the fundamental units **M, L** and **T.**

319. Sudden Impulse.— The impulse of a force which acts for a very short time is called a *sudden impulse.* If the time of action is infinitesimal, the impulse is *instantaneous.*

The value of a sudden impulse will be small unless the force is very great. For $P\Delta t$ is the impulse of a constant force P acting for a time Δt, and this product will be very small if Δt is very small, unless P is very great.

An instantaneous impulse cannot have a finite value unless the

---

* See Art. 219.

force is infinite; impulses which are strictly instantaneous therefore do not come within our experience. There are, however, cases in which a very great force acts for a very short time, and in which the variation of the force within the interval considered cannot be made the subject of observation. Such impulses may often be regarded as instantaneous without important error. Thus, when two bodies come into collision, the force which each exerts upon the other cannot usually be determined, since its time of action is very short; but the value of the impulse may sometimes be found by observing its effect upon the motion.

320. Effect of a Sudden Impulse.— The effect of a sudden impulse is to produce a sudden change in the momentum of the body acted upon. If this change of momentum can be observed, the value of the impulse can be determined, although the force and the time both remain unknown.

321. Ordinary Forces Neglected.— In considering the effect of a sudden impulse, all forces of ordinary magnitude may usually be neglected without important error. Consider the case of a ball struck by a bat. During the impulse the ball is acted upon by the force of gravity (a downward force equal to the weight of the ball). But during the short time of action of the force due to the blow, the change of momentum due to the force of gravity is exceedingly small in comparison with that due to the blow, and may be neglected without appreciable error.

If an impulse is strictly instantaneous, any finite force whatever may be neglected in comparison with the force of the impulse, since the momentum due to a finite force during an infinitesimal time is infinitesimal.

Let P be the force of an instantaneous impulse which produces a finite change of momentum, and let Q be a finite force acting on the particle at the time of the blow. Then the equation of impulse and momentum is

$$\int_{t'}^{t''} P\,dt + \int_{t'}^{t''} Q\,dt = m(v'' - v').$$

If the time of action of the blow, $t'' - t'$, approaches zero, the second term of the first member of this equation approaches zero. Hence Q may be neglected in estimating the change of momentum during the blow.

EXAMPLES.

1. A ball weighing $5\frac{1}{4}$ oz., moving horizontally at the rate of 100 ft.-per-sec., is struck by a bat. Immediately after the blow the ball moves at the rate of 150 ft.-per-sec. in the direction opposite to that of its original motion. What is the value of the impulse?

2. In Ex. 1, if the time occupied by the blow is 0.1 sec., what is the average value of the force (*i. e.*, what constant force would produce the same change of momentum in the same time)?

Ans. 820.3 poundals.

3. What would be the velocity of the ball after the blow, if initially at rest and acted upon by the same impulse as in Ex. 1?

4. If the same blow were applied to a body of 1 lb. mass, what would be its effect?

322. Impact or Collision of Bodies.— When two bodies come into collision, each exerts upon the other at the place of contact a force whose magnitude increases from zero up to some maximum value and then decreases to zero again. The time of action of these forces is so small that their magnitudes cannot be measured; therefore in dealing with cases of impact the discussion must usually be limited to a consideration of the total changes of velocity and momentum produced in the bodies.

Direct and indirect impact.— If two bodies moving in the same straight line collide, their impact is said to be *direct;* their original velocities may have the same direction or opposite directions. If the paths of the two bodies before collision are intersecting lines, the impact is *indirect.* The present discussion will be restricted to the case of direct impact. It will further be assumed that the bodies are of such symmetrical form that the forces acting between them when in contact are directed along their common line of motion.

Elastic and inelastic bodies.— As the two bodies come into collision, each is distorted by the pressure, the amount of distortion increasing as the pressure increases. If the bodies are wholly *inelastic,* the distortion is permanent, the bodies showing no tendency to regain their original forms. In this case the two bodies will move on with a common velocity after the impact. If they are *elastic,* each tends to regain its original form, and in so doing exerts a force upon the other so that the two tend to separate. *Elasticity,* as thus defined, is possessed by bodies in various degrees. The method of specifying the degree of elasticity will be considered below.

323. Law of Action and Reaction Applied to Impact.— If A and B are any two bodies in collision, the forces exerted by them upon each other are at every instant equal and opposite, in accordance with Newton's third law (Art. 259). It follows that the impulses of these forces during any time are equal and opposite. Also, since the momentum due to an impulse is numerically equal to the impulse, the two bodies receive during any part of the time of the collision equal and opposite quantities of momentum. We thus reach the following principle:

The collision of two bodies causes no change in their total momentum.

In the case of direct impact this principle may be stated algebraically as follows: Let m_1, m_2 be the masses of two bodies moving in the same straight line, v_1, v_2 their velocities before collision and v_1', v_2' their velocities at any instant *during* or *after* the collision. (In any particular case algebraic signs must be given to the velocities in accordance with their directions along the line of motion.) Before the collision the total momentum of the two bodies is

$$m_1v_1 + m_2v_2;$$

at a succeeding instant its value is

$$m_1v_1' + m_2v_2'.$$

By the above principle these two quantities are equal, giving the equation

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'. \qquad (1)$$

Equation (1) is not sufficient for the determination of v_1' and v_2'; but if an additional relation between these quantities is known they may be determined. For example, suppose there is some instant during the collision at which the two bodies have the same velocity v; then for that instant $v_1' = v_2' = v$, and (1) becomes

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v,$$

or

$$v = (m_1v_1 + m_2v_2)/(m_1 + m_2). \qquad (2)$$

324. Inelastic Impact.— If two bodies moving in the same line are inelastic, they will not separate after collision (Art. 322); hence in this case equation (2) will give their common velocity after impact.

EXAMPLES.

1. A sphere of mass 10 lbs. moving at the rate of 20 ft.-per-sec. overtakes a sphere of mass 20 lbs. moving at the rate of 10 ft.-per-

sec. If the bodies are inelastic, what is their velocity after the collision? *Ans.* $13\frac{1}{3}$ ft.-per-sec.

2. In Ex. 1, what is the value of the impulse acting on each body during the collision?

Taking the direction in which the bodies are moving as positive, the momentum of the first body changes from 200 to $133\frac{1}{3}$ (F. P. S. units), while the momentum of the second changes from 200 to $266\frac{2}{3}$. Hence the impulse acting upon the first body is $-66\frac{2}{3}$ poundal-seconds, and that acting on the second is $+66\frac{2}{3}$ poundal-seconds.

3. A body whose mass is 5 kilogr., moving at the rate of 700 c.m.-per-sec., meets a body whose mass is 4 kilogr. Both bodies are brought to rest by the collision. Required (*a*) the original velocity of the second body, and (*b*) the value of the impulse acting on each body during the collision.

Ans. (*a*) —875 c.m.-per-sec. (*b*) Two opposite impulses each equal to 3.5×10^6 dyne-seconds.

4. Two bodies collide while moving in opposite directions with velocities inversely proportional to their masses. Show that their total momentum is zero at every instant during the collision. If they are inelastic, what is their common velocity at the end of the collision?

5. A body of 2 lbs. strikes a body of 50 lbs. which is at rest on a smooth horizontal surface. Immediately after the collision the heavier body has a velocity of 4 ft.-per-sec. while the lighter body is at rest. What was the initial velocity of the lighter body? What total impulse acted on each body?

6. Is the solution of Ex. 5 changed by assuming the horizontal surface to be rough? [See Art. 321.]

7. A 2-oz. bullet passes through a block of wood weighing 4 lbs., its velocity being thereby changed from 1,100 ft.-per-sec. to 950 ft.-per-sec. With what velocity does the block move after the impact, if free? What is the value of the total impulse acting on the block?

8. A block weighing 10 lbs. is struck by a 2-oz. bullet which remains imbedded in it. After the collision the block has a velocity of 10 ft.-per-sec. What was the original velocity of the bullet? *Ans.* 810 ft.-per-sec.

325. Elastic Impact.—If two elastic bodies moving in the same straight line collide, it is convenient to divide the time occupied by the impact into two parts: the time T_1 up to the instant of greatest distortion of the bodies, and the time T_2 after that instant. At the instant of greatest distortion the bodies have the same velocity. Before that instant the forces acting between them are resisting

their approach, after that instant the forces are urging their separation.

Let the whole impulse which either body exerts upon the other be divided into two parts; the part acting during T_1 being called the *impulse of compression*, and the part acting during T_2 the *impulse of restitution*. Let the ratio of the latter impulse to the former be called the *coefficient of restitution*, and be represented by e.

Let m_1 and m_2 be the masses of the bodies; v_1 and v_2 their velocities before collision; v their common velocity at the instant of greatest compression; v_1', v_2' their velocities after the collision. Let Q denote the impulse acting upon m_1 during compression, and em_1 the impulse acting on the same body during restitution. The impulses acting on m_2 are then $-Q$ and $-eQ$.

Consider either body, as that whose mass is m_1. Before the collision its momentum is m_1v_1; at the instant of greatest compression it is m_1v; at the end of the collision it is m_1v_1'. Hence

(increase of momentum due to impulse $Q) = m_1(v - v_1)$;
("   "   "   " $eQ) = m_1(v_1' - v)$.

Or, since the impulse is numerically equal to the momentum it produces,

$$Q = m_1(v - v_1); \quad . \quad . \quad . \quad . \quad (3)$$

$$eQ = m_1(v_1' - v). \quad . \quad . \quad . \quad . \quad (4)$$

Eliminating Q and solving for v_1',

$$v_1' = (1 + e)v - ev_1.$$

But by equation (2) (Art. 323),

$$v = (m_1v_1 + m_2v_2)/(m_1 + m_2);$$

which substituted in the value of v_1' gives, after reducing,

$$v_1' = v_1 - \frac{m_2}{m_1 + m_2}(1 + e)(v_1 - v_2). \quad . \quad . \quad (5)$$

In like manner, by writing equations similar to (3) and (4) for the other body, we may find

$$v_2' = v_2 - \frac{m_1}{m_1 + m_2}(1 + e)(v_2 - v_1). \quad . \quad . \quad (6)$$

Equations (5) and (6) determine the velocities of the bodies after they separate. The value of the impulse Q and of the total impulse

$(1+e)Q$ may be determined from the equation of impulse and momentum. Thus, the total change of momentum of the body m_1 is $m_1(v_1' - v_1)$. Equating this to the total impulse, and using the value of v_1' given by equation (5),

$$(1+e)Q = -\frac{m_1 m_2}{m_1 + m_2}(1+e)(v_1 - v_2); \qquad (7)$$

$$Q = -\frac{m_1 m_2}{m_1 + m_2}(v_1 - v_2). \qquad (8)$$

Equation (7) gives the whole impulse acting upon the body m_1. Assuming v_1 greater than v_2, the impulse is negative. Equation (8) states that the impulse up to the instant of greatest compression is independent of e, that is, independent of the degree of elasticity.

The value of e may lie anywhere between 0 and 1. For inelastic impact $e = 0$, and the value of v_1' is

$$v_1' = (m_1 v_1 + m_2 v_2)/(m_1 + m_2), \qquad (9)$$

as in Art. 324. For perfect restitution, $e = 1$, and

$$v_1' = v_1 - \frac{2m_2}{m_1 + m_2}(v_1 - v_2); \qquad (10)$$

$$v_2' = v_2 - \frac{2m_1}{m_1 + m_2}(v_2 - v_1). \qquad (11)$$

Examples.

1. A man weighing 160 lbs. leaped into a boat whose mass was 100 lbs., thereby causing it to move from rest with a velocity of 10 ft.-per-sec. With what velocity did the man leap?

Ans. $16\frac{1}{4}$ ft.-per-sec.

2. A boat weighing 200 lbs. is at rest in still water when a person weighing 150 lbs. starts to move from one end to the other. If he moves 10 ft. relatively to the boat, how far does he move relatively to the water, and how far does the boat move?

By Newton's third law, the force urging the man forward is at every instant equal to the force urging the boat in the opposite direction. Let P be the magnitude of each of these forces in poundals; then at any instant the acceleration of the boat is $P/200$ and that of the man $P/150$ in the opposite direction. The velocities due to these accelerations in any time are in the same ratio as the accelerations; and the distances described by the two bodies in the same time are proportional to their velocities, and therefore to their accelerations. The boat thus moves three-fourths as far as the man.

3. Two perfectly elastic bodies moving along the same line collide. Prove that they exchange velocities.

4. A body whose mass is 5 kilogr., moving at the rate of 5 met.-per-sec., collides with a body whose mass is 4 kilogr., moving in the same direction at the rate of 4 met.-per-sec. The coefficient of restitution is 0.5. Determine the velocities of the bodies after they separate. *Ans.* $4\frac{1}{3}$ met.-per-sec. and $4\frac{5}{6}$ met.-per-sec.

5. In Ex. 4, compute the value of the impulse acting during compression, and of the total impulse during the collision.
Ans. $(2/9) \times 10^6$ dyne-seconds; $(1/3) \times 10^6$ dyne-seconds.

6. A body weighing 10 lbs., moving at the rate of 12 ft.-per-sec., overtakes a body weighing 18 lbs., moving at the rate of 5 ft.-per-sec. After the collision the velocity of the former body is 6 ft.-per-sec. Required (*a*) the final velocity of the second body; (*b*) the coefficient of restitution. *Ans.* (*a*) $8\frac{1}{3}$ ft.-per-sec. (*b*) $\frac{1}{3}$.

7. Prove that, in any case of direct collision, the ratio of the relative velocity of the two bodies after impact to their relative velocity before impact is numerically equal to the coefficient of restitution.

8. A rifle weighing 3 lbs. is discharged while lying on a smooth horizontal plane. The weight of the bullet is 2 oz. and it leaves the barrel with a velocity of 1,400 ft.-per-sec. What will be the velocity of the "kick"? What is the impulse of the kick?

9. If the discharge of a rifle were strictly instantaneous, what would be the force of the kick?

10. A man weighing 165 lbs. leaps from a boat weighing 124 lbs. into a boat weighing 150 lbs. If the boats are initially at rest, compare their velocities after the leap.

§ 2. *Motion in Any Path.*

326. Momentum a Vector Quantity.—The momentum of a moving particle has been defined (Art. 312) as a quantity proportional directly to its mass and to its velocity. When motion in a curved path is studied, it must be remembered that *direction* as well as magnitude is an essential part of the value of velocity, and therefore of momentum. Momentum is, in fact, a vector quantity, its direction coinciding at every instant with that of the velocity of the particle.

327. Moment of Momentum or Angular Momentum.—The momentum of a particle is at every instant associated with a certain *position.* It is, in fact, a *localized vector quantity* (Art. 26), its position-line at any instant being the straight line along which the motion

is directed. The position-line is the tangent to the path at the instantaneous position of the particle.

The *moment of momentum* of a particle with respect to any point in the plane of the motion is the product of the momentum into the perpendicular distance of its position-line from the origin of moments. Moment of momentum is also called *angular momentum.*

328. Increment of Momentum.— If the velocity of a particle varies (either in direction or in magnitude or in both) so also does the momentum. The *increment of momentum* for any interval may be found in the manner already described for finding increment of velocity (Art. 246) or of any variable vector (Art. 22). Thus, let AB (Fig. 142) be the path described by a particle during any interval, v_1 being the magnitude of the velocity in the position A and v_2 its magnitude in the position B. At the beginning of the interval the momentum has the value mv_1, directed along the tangent to the path at A; at the end of the interval its value is mv_2, directed along the tangent at B. Representing these values by vectors OA' and OB', the vector $A'B'$ represents the increment of momentum for the given interval.

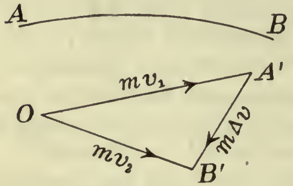


FIG. 142.

329. Acceleration of Momentum a Vector Quantity.— The *acceleration of momentum* (or *mass-acceleration*) of a particle is a vector quantity proportional directly to the mass and to the acceleration. Its direction is that of the acceleration; it does not, in general, coincide with that of the tangent to the path at the position of the particle.

Acceleration of momentum bears the same relation to momentum that acceleration bears to velocity. It may be defined concisely as *momentum-increment per unit time;* just as acceleration is defined as *velocity-increment per unit time* (Art. 247).

EXAMPLES.

1. If the velocity of a particle of mass 18 lbs. changes from 12 ft.-per-sec. in a certain direction to 20 ft.-per-sec. in the opposite direction, what is the momentum-increment?

Ans. 576 F. P. S. units.

2. If the velocity of a particle of 18 lbs. mass changes from 12 ft.-per-sec. in a horizontal direction to 20 ft.-per-sec. vertically downward, what are the magnitude and direction of the momentum-increment? *Ans.* 419.9 F. P. S. units ; 30° 58′ from vertical.

3. A particle of 60 lbs. mass describes a circle of 6 ft. radius at the uniform rate of 120 revolutions per minute. Determine the magnitude and direction of the increment of momentum received by the particle while making (*a*) one-sixth of a revolution, (*b*) one-fourth of a revolution, and (*c*) one-half a revolution.

Ans. (*a*) 4,524 F. P. S. units.

4. A particle of 20 gr. mass describes a circle of 90 c.m. radius with a uniform speed of 250 c.m.-per-sec. Determine completely (in magnitude and direction) the increment of momentum in one-hundredth of a second.

Ans. Its magnitude is 139 C. G. S. units, nearly.

5. With the data of Ex. 4, compute the moment of the momentum (*a*) about the center of the circle; (*b*) about a point in the circumference. (*c*) Compute the greatest and least values of the moment of the momentum about a point 40 c.m. from the center.

330. Impulse of a Force Whose Direction Varies.— If the direction of a force varies, this variation must be taken into account in computing the impulse. Impulse must now be regarded as a vector quantity, its direction being determined by that of the force; and the impulses for successive intervals of time are to be combined by vector addition. Thus, if P_1' and P_2' denote the values of the impulse for two successive intervals Δt_1 and Δt_2, the impulse for the entire interval $\Delta t_1 + \Delta t_2$ is the vector sum of P_1' and P_2'. If the force has a constant direction during the interval Δt_1, and a constant but different direction during the interval Δt_2, P_1' and P_2' may each be computed as in Art. 315 or Art. 316. But if the direction of the force varies continuously, a different method must be employed.

Let it be required to compute the impulse of any variable force P during the interval from t' to t''. Let the interval be subdivided into small parts Δt_1, Δt_2, ... , and let the values of P at the beginnings of these intervals be P_1, P_2, ... Approximate values of the impulses for the successive small intervals are

$P_1 \Delta t_1$ in direction of P_1,

$P_2 \Delta t_2$ in direction of P_2,

. . . . ;

and the required total impulse P' is approximately equal to the

vector sum of $P_1\Delta t_1$, $P_2\Delta t_2$, ... The approximation may be made as close as desired by taking the intervals sufficiently small. The exact value of the total impulse is the limit approached by the approximate value as Δt_1, Δt_2, . . . all approach zero. That is,

$$P' = \text{limit } [P_1\Delta t_1 + P_2\Delta t_2 + \quad . \quad . \quad . \quad],$$

it being understood that the + sign denotes vector addition.*

It will be noticed that the above process of computing impulse is independent of the path of the particle upon which the force acts. So far as its magnitude and direction are concerned, impulse depends only upon *force* and *time*.

331. Vector Equation of Impulse and Momentum.—The principle of impulse and momentum, already stated for the case of rectilinear motion, may now be extended to the case of a particle describing any path under the action of a force varying in any manner.

The impulse of the force during any interval is equal in magnitude and direction to the change of momentum of the particle during that interval.

This statement includes the proposition given in Art. 317. The equation

$$\text{impulse} = \text{momentum-increment}$$

must now be interpreted as a vector equation ; in the special case of rectilinear motion it reduces to an algebraic equation.

332. Algebraic Computation of Impulse of Variable Force.—Usually the simplest method of computing the value of the impulse of a force whose direction varies is to resolve the force into components parallel to fixed rectangular axes and determine the impulse of each axial component separately. Restricting the discussion to the case in which the path of the particle and the line of action of the force lie in a plane, let P be the value of the force at time t, and let the axial components of P be X and Y. Let P' denote the impulse of P for the interval from t' to t'', X' and Y' being the axial components of P'. Then

$$X' = \int_{t'}^{t''} X\,dt; \quad Y' = \int_{t'}^{t''} Y\,dt.$$

* The above reasoning may be expressed in the language of differentials, as follows: Let P be a vector symbol denoting the value of the force at the instant t, and P' a vector symbol denoting the value of the required impulse. Then $dP' = P\,dt$. This is a differential equation, both members of which are vectors.

From the values of X' and Y' the magnitude and direction of P' may be determined.

333. Algebraic Equations of Impulse and Momentum.— In the case of plane motion the vector equation of impulse and momentum is equivalent to two algebraic equations. These may be deduced immediately from the differential equations of motion (Art. 287). These equations are

$$X = m\ddot{x}, \quad Y = m\ddot{y}.$$

Multiplying each through by dt and integrating between limits t' and t'', the values of $\dot{x}$ and $\dot{y}$ at the limits being $\dot{x}'$, $\dot{x}''$ and $\dot{y}'$, $\dot{y}''$ respectively,

$$\int_{t'}^{t''} X\,dt = m(\dot{x}'' - \dot{x}')\,; \quad \int_{t'}^{t''} Y\,dt = m(\dot{y}'' - \dot{y}').$$

The first members of these equations are the axial components of the impulse of the resultant force acting upon the particle (Art. 332); the second members are the axial components of the increment of momentum of the particle. Together the equations express the proposition that *the resultant impulse is equal in magnitude and direction to the total increment of momentum.* The general principle of Art. 331 is thus an immediate consequence of the fundamental equations of motion of a particle.

334. Impulse and Momentum as Localized Vector Quantities.— In the foregoing discussion no reference has been made to the path of the particle nor to the position in space of the line of action of the force. These must, however, be taken into account in a complete explanation of the principle of impulse and momentum.

The momentum of a particle is at every instant directed along a definite line in space. This may be regarded as its position-line, momentum being thus a *localized vector quantity* (Art. 26).

A force acting upon a particle has at every instant a definite line of action. The impulse of a force whose line of action remains constant may therefore be regarded as a localized vector quantity whose position-line is the line of action of the force. If the line of action of the force varies, the elementary impulse $P\,dt$ corresponding to an elementary time dt may be regarded as a localized vector quantity whose position-line coincides with the instantaneous position of the line of action of the force.

335. Angular Impulse. — Let P denote the value of a force at any instant t, and P' the impulse during the time from t' to t''. Choosing any origin of moments O (Fig. 143), let a be the perpendicular distance from O to the line of action P. Then Pa is the moment of the force with respect to O. During a small interval Δt let the particle to which the force is applied move a short distance AB. The change in P will be small; the impulse during the small interval will be approximately equal to $P\Delta t$; and the moment of the impulse will differ little from $Pa\,\Delta t$.

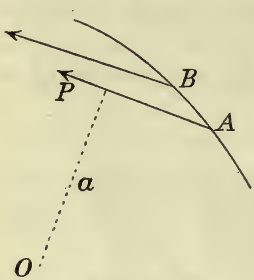


Fig. 143.

Let the whole time from t' to t'' be subdivided into small parts Δt_1, Δt_2, . . . , and let the values of P at the beginnings of these small intervals be P_1, P_2, ... , the corresponding values of the arm a being a_1, a_2, ... The sum

$$P_1 a_1 \Delta t_1 + P_2 a_2 \Delta t_2 + \quad . \quad . \quad .$$

may be regarded as an approximate value of the moment of the resultant impulse P'. Taking the intervals smaller and smaller, approaching zero, the exact value of the moment of the impulse is

$$\text{limit}\ [P_1 a_1 \Delta t_1 + P_2 a_2 \Delta t_2 + \quad . \quad . \quad . \quad] = \int_{t'}^{t''} Pa\,dt.$$

If the moment of the force with respect to O is represented by G,

$$G = Pa,$$

and the moment of the impulse is equal to

$$\int_{t'}^{t''} G\,dt.$$

The moment of the impulse of a force about a given point is also called its *angular impulse* about that point.

336. Position-Line of Resultant Impulse.— It has been pointed out that the impulse of a force during an elementary time dt may be regarded as having a definite position-line, even when the line of action of the force is variable. The resultant impulse for an interval of any length may be regarded as localized in a line whose position

is determined by the value of the moment of the impulse about any given point. Its distance a' from the origin of moments must satisfy the equation

$$P'a' = \int_{t'}^{t''} Pa\,dt.$$

It will be observed that the method of determining the magnitude, direction and position-line of the resultant impulse for any interval is essentially the same as that of finding the magnitude, direction and line of action of the resultant of any system of forces acting in the same plane on a rigid body (Art. 97). In the case of forces, the resultant is equal to the vector sum of the components; and the moment of the resultant is equal to the algebraic sum of the moments of the components. In the case of the impulse, the resultant impulse for any interval is equal to the vector sum of the impulses for the elementary intervals; and the moment of the resultant impulse is equal to the algebraic sum of the moments of the elementary impulses. The similarity of these relations may be further illustrated.

Let AB (Fig. 144) represent the path described by the particle during the time from t' to t''. Let the whole time be subdivided into small intervals Δt_1, Δt_2, . . . ; let A, a, b, . . . be the positions of the particle at the beginnings of these intervals; and let P_1, P_2, . . . be the corresponding values of the force, their directions being as shown in the figure The impulses during Δt_1, Δt_2, . . . have approximately the values

$P_1 \Delta t_1$ along the line AA',

$P_2 \Delta t_2$ along the line aa',

. . . .

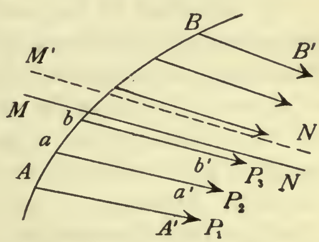


FIG. 144.

Let these be combined as if they were forces; their resultant is equal to their vector sum and acts along some determinate line, as $M'N'$; and this resultant is an approximate value of the resultant impulse P'. The exact value is the limit approached by the approximate value as Δt_1, Δt_2, . . . are all made to approach zero. The line $M'N'$ approaches a limiting position MN, which is the position-line of the resultant impulse P'.

EXAMPLES.

1. A particle is projected horizontally with a given velocity V, after which it is acted upon by no force except gravity. Determine the position-line of the resultant impulse of the force of gravity during any time.

Let the particle be projected from the point A (Fig. 145) in the direction AB', and let AB be the path described during the time T. Let this time be subdivided into a number of small intervals, each equal to Δt, and let $A, a, b, \ldots$ be the positions of the particle at the beginnings of these intervals. If m is the mass, the force is equal to mg and is constant in direction. Applying the method above described, the required impulse is approximately equal to the resultant of a series of equal impulses $mg\Delta t$ acting in lines AA', aa', bb', ... The successive distances between these lines are equal, since the horizontal velocity is constant; hence the position-line of the resultant is midway between the extreme lines AA', gg'. The limiting position of this line, as Δt is made to approach zero, bisects AB'. The magnitude of the resultant impulse is obviously mgT.

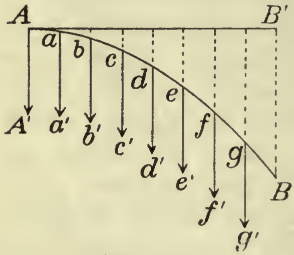


FIG. 145.

2. In Ex. 1, compute the moment of the resultant impulse by integration, and thus verify the above result.

337. Equation of Angular Impulse and Angular Momentum.— The impulse of a force and the increment of momentum produced by it, during any element of time, are not only equal in magnitude and direction but have the same position-line. From this it follows that their moments about any origin are equal. And since this equality holds for every element of time, it holds for any interval whatever. That is,

The angular impulse of a force for any interval of time is equal to the angular momentum produced by the force during that time.

This equation and the two given in Art. 332 are the complete algebraic expression of the principle of impulse and momentum as applied to the motion of a particle in a plane. To summarize:

The proposition that impulse is equal to change of momentum, in its full meaning, must be understood to express *equality of magnitude*, *agreement in direction*, and *identity of position-lines*. In case of plane motion, this is completely expressed by three algebraic

equations. Two of these may be obtained by resolving in any two directions, the third by taking moments about any point.

338. Geometrical Illustration of General Equation of Impulse and Momentum.— Let the curve AB (Fig. 146) be the path described by a particle during the interval from t' to t''. Let v' and v'' be the initial and final values of the velocity. From any point O draw vectors OA', OB', representing the initial and final values of the momentum. These vectors are parallel to the tangents to the path at A and at B respectively, and $OA' = mv'$, $OB' = mv''$. The increment of momentum $m\Delta v$ is then represented by the vector $A'B'$. To apply the principle of impulse and momentum, the position-line of the momentum-increment must be determined.

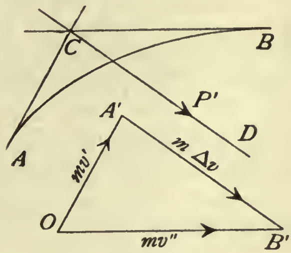


FIG. 146.

The position-line of the initial momentum is the tangent to AB at A; that of the final momentum is the tangent to AB at B; the position-line of the momentum-increment therefore passes through C, the intersection of these tangents.

Let P' be the resultant impulse applied to the particle during the interval; then $P' = m\Delta v$, and its position-line is CD, parallel to $A'B'$.

In the foregoing explanation of the general principle of impulse and momentum, it has been assumed that each of these quantities is subject to the laws of composition and resolution which apply to forces acting upon a rigid body. As applied to impulse, these laws have been explicitly stated in Art. 336. As applied to momentum, it is assumed that the increments of momentum received by a particle during successive intervals are to be combined as if they were forces acting on a rigid body in order to produce the momentum-increment for the entire interval.

These are arbitrary assumptions, amounting in reality to *definitions* of resultant impulse and resultant momentum. They are extremely useful because they make it possible to state very concisely the full meaning of the equation of impulse and momentum. These definitions are especially valuable when the general principle is extended to systems of particles and to rigid bodies.

EXAMPLES.

1. In Ex. 1, Art. 336, show that the tangent to the path at B bisects AB'. [Apply the general principle of impulse and momentum, using the result already found in the solution of the example.]

2. If A and B are any two points in the path of a projectile, show by the principle of impulse and momentum that the tangents at A and B and the vertical line bisecting the chord AB intersect in a point. [The position-line of the resultant impulse may be located as in Ex. 1, Art. 336.]

3. A particle describes a circle at a uniform rate. Show that the resultant impulse applied to it during any interval passes through the center. Show also that its position-line bisects the path described during the interval.

339. Sudden Impulse. — If a particle receives a sudden impulse whose direction is inclined to that in which the particle is moving, the direction of motion is suddenly changed. The relation between the initial and final velocities and the impulse is expressed by the vector equation of impulse and momentum, irrespective of the time occupied by the impulse.

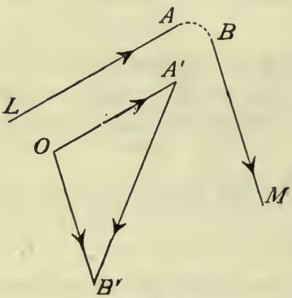


FIG. 147.

During the time of action of a sudden impulse the particle moves over a very short distance. The direction of the motion may, however, change by any angle. In general, therefore, a sudden impulse causes a rapid curvature of the path. Let a particle describing the straight path LA (Fig. 147) receive a blow whose total impulse is represented by the vector $A'B'$. If OA' represents the initial momentum, OB' will represent the momentum after the blow. The path after the blow will be some line BM, parallel to OB'. The path during the impulse will be some curve AB. The shorter the time occupied by the blow, the shorter the path AB. If the impulse were instantaneous, A and B would fall together, as in Fig. 148.

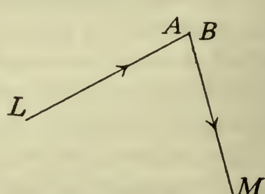


FIG. 148.

Examples.

1. A ball whose mass is $5\frac{1}{4}$ oz. is moving horizontally at the rate of 100 ft.-per-sec. when it is struck by a bat in such a way that immediately after the blow it has a velocity of 150 ft.-per-sec. in a direction making an angle of 30° upward from the horizontal. Required the value of the impulse.

Ans. 79.34 poundal-seconds, inclined 18°4′ upward from the horizontal, assuming the horizontal velocity to be reversed in direction by the blow.

2. In Ex. 1, suppose the ball to receive the same blow when moving at an angle of 10° downward from the horizontal. How will it move immediately after the blow?

3. In the same case, if the blow occupies 0.01 sec., what is the average value of the force? What will be the effect of gravity on the motion of the ball during the blow?

340. Collision of Bodies Moving in Different Paths.— The collision of bodies moving in different paths presents a less simple problem than that treated in Arts. 322–325. No discussion of this problem will be given here, except to point out the application of the law of action and reaction.

As a consequence of this law, the total impulses acting on two bodies during their collision are equal and opposite. It follows that their momentum-increments are equal and opposite. The vector sum of their momenta therefore has the same value after collision as before. Its value is, in fact, the same at every instant during the collision.

Examples.

1. A body of 5 lbs. mass strikes a fixed surface which deflects it 20° and changes its speed from 25 ft.-per-sec. to 10 ft.-per-sec. What is the total impulse exerted upon the body during the collision?

Ans. 79.8 poundal-seconds, directed at angle 167° 38′ with the original motion.

2. A ball A of 6 lbs. mass is at rest on a horizontal plane when it is struck by a ball B of 4 lbs. mass moving at the rate of 12 ft.-per-sec. After the collision A is moving at the rate of 4 ft.-per-sec. in a direction inclined 60° to the original velocity of B. How does B move after the collision?

Ans. With velocity 10.4 ft.-per-sec. at right angles to motion of A.

3. Two bodies moving in opposite directions with velocities inversely proportional to their masses collide in such a way as to be deflected from the original directions of motion. Show that after the collision they move in opposite directions with velocities inversely proportional to their masses.

CHAPTER XVII.

WORK AND ENERGY.

§ 1. *Work in Case of Rectilinear Motion.*

341. **When a Force Does Work.**—A force is said to *do work upon the body* to which it is applied when its point of application moves in the direction toward which the force acts.

342. **Quantity of Work Done by Constant Force.**—The amount of work done by a constant force is a quantity proportional directly to the magnitude of the force and to the distance moved by its point of application in the direction of action of the force. The numerical value of the work depends upon the unit in which it is expressed.

The *unit of work* usually chosen is the quantity of work done by the unit force when its point of application moves the unit distance in the direction of action of the force. If the pound is taken as the unit force and the foot as the unit distance, the unit work is the quantity of work done by a force of one pound when its point of application moves one foot in the direction of action of the force. This unit is called a *foot-pound*, and is the unit commonly employed by engineers. In the C. G. S. system the unit of work is the quantity of work done by a force of one dyne while its point of application moves one centimeter; the name *erg* has been given to this unit.* In the F. P. S. system the unit of work is the *foot-poundal.*

When the point of application of a force moves a certain distance in the direction of action of the force, the force is said to *act through* that distance. The unit work may therefore be defined as *the quantity of work done by the unit force acting through the unit distance.*

---

* The erg is very small in comparison with the foot-poundal or the foot-pound. Thus,

$$1 \text{ foot-poundal} = 421{,}390 \text{ ergs};$$

$$1 \text{ foot-pound} = 1.356 \times 10^7 \text{ ergs}.$$

(The last result assumes $g = 981$ C. G. S. units.)

For practical use, especially in electrical engineering applications, another unit of work, the *joule*, has been introduced, defined as equal to 10^7 ergs.

343. Positive and Negative Work.—If the point of application of a force moves in a direction opposite to that of the force, the force is said to do negative work upon the body.

If P is the algebraic value of a constant force, and Δx that of the displacement of its point of application, the positive direction being the same for both, the value of the work is given with proper sign by the product

$$P\Delta x.$$

EXAMPLES.

1. A body weighing 10 lbs. is thrown upward against gravity. Compute the work done upon it by its weight (a) while it rises 10 ft. and (b) while it falls 10 ft.

Ans. (a) $-100g$ foot-poundals. (b) $+100g$ foot-poundals.

2. If the resistance of the air amounts to a constant force of 2 lbs., compute the work done by it in both cases of Ex. 1.

Ans. —20 foot-pounds in each case.

344. Work Done by Variable Force.—If the value of a force varies during the displacement of its point of application, the work must be computed by integration.

Let OB (Fig. 149) be the path described by the point of application (also the line of action of the force). Let P be the value of the force when the point of application is at distance x from O, and let it be required to compute the work done by P while the point of application moves from A to B.

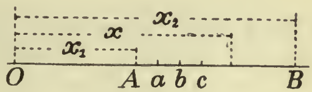


FIG. 149.

Divide AB into small parts $Aa = \Delta x_1$, $ab = \Delta x_2$, $bc = \Delta x_3$, . . . and let P_1, P_2, P_3, . . . be the values of P when the point of application is at A, a, b, ... Let W represent the required work; then, approximately,

$$W = P_1\Delta x_1 + P_2\Delta x_2 + P_3\Delta x_3 + \quad . \quad . \quad . \quad .$$

By taking Δx_1, Δx_2, . . . small enough, the approximation may be made as close as desired. The exact value of the work is the limit approached by the approximate value as Δx_1, Δx_2, . . . are made to approach zero. That is, if $x_1 = OA$ and $x_2 = OB$,

$$W = \text{limit}\,[P_1\Delta x_1 + P_2\Delta x_2 + \quad . \quad . \quad . \quad] = \int_{x_1}^{x_2} P\,dx.$$

345. Graphical Representation of Work Done by Variable Force.— At every point of the line AB representing the displacement of the point of application (Fig. 150), erect an ordinate whose length is equal to the value of P for that position; the extremities of these ordinates lie on a curve $A'B'$. The area bounded by the curve, the line AB, and the ordinates AA', BB', is equal to the work done by P during the displacement AB. For, the expression above found for the work,

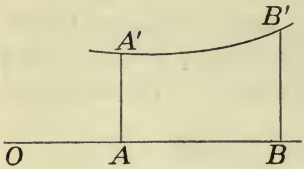


Fig. 150.

$$\int_{x_1}^{x_2} P\,dx,$$

is also the value of the area.

Let the work-diagram be drawn in each of the following examples.

Examples.

1. A body is suspended by an elastic string whose unstretched length is 4 ft. Under a pull of 10 lbs. the string stretches to a length of 5 ft. Required the work done on the body by the tension of the string while its length changes from 6 ft. to 4 ft.

Ans. 20 foot-pounds.

2. A body whose mass is m falls vertically to the earth's surface from a height equal to the radius R. Compute the work done by the earth's attraction during the fall. *Ans.* $\frac{1}{2}mgR$ kinetic units.

3. In Ex. 2, let $m = 100$ lbs., $R = 21{,}000{,}000$ ft. Give the value of the work in foot-pounds. *Ans.* 1.05×10^9.

4. A particle moves in a straight line under the action of a force directed toward a fixed point in the line of motion and varying directly as the distance from that point. Compute the work done on the particle during a given displacement.

Ans. Let P' be the value of the force when the particle is at the unit distance from the fixed point, x_1 and x_2 the initial and final distances from that point. Then the required work is $\frac{1}{2}P'(x_1^2 - x_2^2)$.

346. Dimensions of Unit Work.—If the units of force and length are both chosen arbitrarily, the unit work has dimensions **FL**. But in the kinetic system (Art. 219), $\mathbf{F} = \mathbf{ML/T^2}$, which reduces the unit work to the dimensions

$$\mathbf{ML^2/T^2}.$$

§ 2. *Work in Any Motion of a Particle.*

347. **General Definition of Work.**— When the point of application of a force receives a displacement which is not along the line of action, the definition of work must be enlarged.

Let a force P remain constant in magnitude and direction while its point of application describes any straight line AB (Fig. 151). The quantity of work done depends upon the value of AB', the orthographic projection of AB upon the line of action of P. This projection may be called the *effective displacement.*

The quantity of work done by a constant force while its point of application receives any rectilinear displacement is equal to the product of the force into the effective displacement.

If the direction of the effective displacement coincides with that of the force, the work is positive; if the effective displacement is opposite to the force the work is negative. If the actual displacement

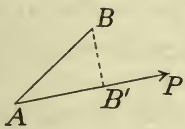


FIG. 151.

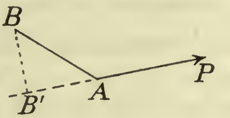


FIG. 152.

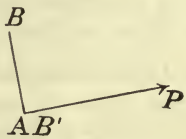


FIG. 153.

is at right angles to the direction of the force, the effective displacement is zero. These three cases are illustrated in Figs. 151, 152, 153.

The cases of positive and negative work may be illustrated by a body sliding along an inclined plane. Let P be the weight of the body. In computing the work done by the force P, the effective displacement is the vertical projection of the distance moved by the body. If the body descends a vertical distance h, the work is $+Ph$. If it rises a vertical distance h, the work is $-Ph$. If the plane is horizontal, the work is zero.

348. **Work Done by Constant Force Whose Point of Application Receives Any Displacement.**— The above rule for computing the work of a constant force may be extended to the case in which the displacement follows any curve AB. It may be shown that the quantity of work done by the force is equal to the product of the

magnitude of the force into the projection of AB upon a line parallel to the force. For, suppose the path to be replaced by a broken line $AabB$ (Fig. 154), a and b being any points of the curve. The work done during the displacement along this broken line is equal to

$$P(Aa' + a'b' + b'B') = P \times AB'.$$

It is evident that the same value of the work results if the displacement follows any broken line whatever joining A and B. If the number of points such as a and b is taken greater and greater, the broken line approaches the curve as a limit; hence the work done during the motion along the curve has the value $P \times AB'$.

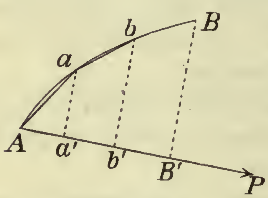


FIG. 154.

It follows that the work done by a force which remains constant in magnitude and direction while its point of application moves from A to B has the same value whatever the path described between A and B. An example of this is the work done by gravity during the movement of bodies near the earth's surface.

349. General Algebraic Formula for Work of Constant Force.— The general rule for computing the work done by a force P which remains constant in magnitude and direction while its point of application moves from A to B along any path may be stated algebraically in the form

$$P \times (AB \cos \theta),$$

in which θ is the angle between the straight line AB and the direction of P. The work is positive or negative, according as the angle θ is less or greater than 90° (it may always be so measured as not to exceed 180°).

Since $P \times (AB \cos \theta) = (P \cos \theta) \times AB$, it is evident that *the work done is equal to the product of the displacement into the resolved part of the force in the direction of the displacement.* It is useful to remember the rule in both forms.

350. Work Done by Variable Force During Any Displacement of Its Point of Application.— Let a force P vary in magnitude and direction while the particle to which it is applied describes any curved path AB (Fig. 155). On AB choose any number of points $a, b, c,$ . . . . , dividing the curve into small parts. Let $P_1, P_2,$ . . . be the values of the magnitude P when the par-

ticle is at A, a, . . . ; and let $\theta_1 =$ angle between P_1 and chord Aa, $\theta_2 =$ angle between P_2 and chord ab, etc. Then an approximate value of the work done by the force is

$$P_1 \cos \theta_1 (Aa) + P_2 \cos \theta_2 (ab) + P_3 \cos \theta_3 (bc) + \quad . \quad . \quad . \quad .$$

The smaller the parts Aa, ab, bc, etc., the closer the approximation. The true value of the work is the limit of the above quantity as the number of points a, b, c, etc., is increased indefinitely. That is, if W is the true value of the work,

$$W = \text{limit } [(P_1 \cos \theta_1) \times Aa + (P_2 \cos \theta_2) \times ab + \quad . \quad . \quad . \quad]$$
$$= \int (P \cos \theta) ds.$$

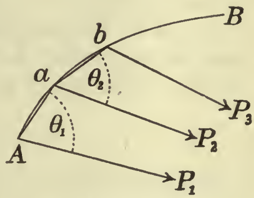


FIG. 155.

Here θ represents the angle between the tangent to the curve at any point and the corresponding direction of P, s is the length of the curve from some fixed point to the position of the particle, and the limits of the integration must be so taken as to include the entire displacement AB. If s' and s'' are the values of s at A and at B respectively,

$$W = \int_{s'}^{s''} (P \cos \theta) \, ds.$$

351. Work of Central Force.—If the force is always directed toward a fixed point, the above expression for the work may be reduced to a convenient form as follows:

Let O (Fig. 156) be the center of force, and AB the path. Let r be the radius vector of the particle measured from O, r' and r'' being the values of r at A and B. Let s be measured positively in the direction AB. Drawing MN tangent to the path, θ is the angle OMN. Evidently,

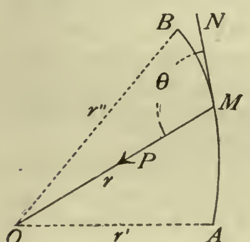


FIG. 156.

$$dr = ds \cos (\pi - \theta) = - ds \cos \theta.$$

The above value of the work therefore reduces to

$$W = - \int_{r'}^{r''} P dr.$$

If P is a known function of r, W may be found by direct integration.

This formula shows that if a particle moves under the action of a central force which is any function of the distance from the center, the work done by the force during any motion depends only upon the initial and final distances from the center.

EXAMPLES.

1. Compute the work done on a particle by an attractive force varying according to the law of gravitation.

Let P be the magnitude of the force at any distance r from the center of attraction, and let P_1 be the value of P when $r = r_1$. According to the law of gravitation we have

$$P : P_1 = r_1^2 : r^2;$$

or

$$P = P_1 r_1^2/r^2 = k/r^2,$$

k being a constant. If the body moves so that the initial and final values of r are r' and r'', the work done is

$$W = -\int_{r'}^{r''} P\,dr = -\int_{r'}^{r''} \frac{k\,dr}{r^2} = k\left(\frac{1}{r''} - \frac{1}{r'}\right).$$

2. In Ex. 1, let the earth be the attracting body.

In this case, if G denotes the weight of the body at the earth's surface, and R the radius of the earth, we have $k = GR^2$; hence

$$W = GR^2(1/r'' - 1/r').$$

3. If a body weighs 1,000 lbs. at the earth's surface, how much work will be done upon it by the earth's attraction while it falls to the surface from an elevation of 10,000,000 ft.? (Take $R =$ 21,000,000 ft.)

4. A particle moves under the action of a central force varying directly as the distance from the center. Compute the work done by the force during any motion.

[Since the work done depends only upon the initial and final distances from the center of attraction, it is evident that it has the same value as in Ex. 4, Art. 345.]

5. A body describes a straight line while acted upon by a force of constant magnitude P lbs. directed always toward a fixed point distant h ft. from the line. Determine the work done by the force while the body passes between any two given positions.

6. In Ex. 5, let $h = 10$ ft., $P = 14$ lbs. Compute the work done by the force while the body moves 40 ft., starting from its nearest position to the fixed point. *Ans.* —437 foot-pounds.

352. Work Done by Resultant of Any Number of Forces.— *The quantity of work done by the resultant of any number of concurrent forces during any displacement of their point of application*

is equal to the algebraic sum of the quantities of work done by the several forces during the same displacement.

Let P be the resultant; $P_1, P_2, \ldots$, the several forces; $\theta, \theta_1, \theta_2, \ldots$, the angles between $P, P_1, P_2, \ldots$ and the tangent to the path of the particle. Since the resolved part of P in any direction is equal to the algebraic sum of the resolved parts of $P_1, P_2, \ldots$ in that direction,

$$P \cos \theta = P_1 \cos \theta_1 + P_2 \cos \theta_2 + \ldots .$$

Hence $(P \cos \theta)ds = (P_1 \cos \theta_1)ds + (P_2 \cos \theta_2)ds + \ldots$; and

$$\int_{s'}^{s''} (P \cos \theta)ds = \int_{s'}^{s''} (P_1 \cos \theta_1)ds + \int_{s'}^{s''} (P_2 \cos \theta_2)ds + \ldots .$$

The first member of this equation represents the work done by P, while the successive terms of the second member represent the quantities of work done by $P_1, P_2, \ldots$, respectively (Art. 350). Hence the proposition is proved.

Examples.

1. A body weighing 50 lbs. slides a distance of 8 ft. down a plane inclined 20° to the horizontal, against a constant retarding force of 4 lbs. due to friction. Compute the total work done upon the body by its weight and the friction. *Ans.* 104.8 foot-pounds.

2. Prove that the total work done by gravity upon a system of particles during any motion has the same value as if the entire mass were concentrated at the center of gravity.

Let the system consist of particles whose weights are $W_1, W_2, \ldots$ and whose elevations above a certain horizontal plane are initially $z_1, z_2, \ldots$; and let their elevations above the same plane in the final positions be $z_1', z_2', \ldots$. The total work done by gravity upon all the particles is

$$W_1(z_1 - z_1') + W_2(z_2 - z_2') + \ldots$$
$$= (W_1 z_1 + W_2 z_2 + \ldots) - (W_1 z_1' + W_2 z_2' + \ldots).$$

But if the elevation of the center of gravity changes from $\bar{z}$ to $\bar{z}'$,

$$W_1 z_1 + W_2 z_2 + \ldots = (W_1 + W_2 + \ldots)\bar{z};$$
$$W_1 z_1' + W_2 z_2' + \ldots = (W_1 + W_2 + \ldots)\bar{z}';$$

and the above value of the work is therefore equal to

$$(W_1 + W_2 + \ldots)(\bar{z} - \bar{z}'),$$

which is the same as the work which would be done by gravity on the whole mass if concentrated at the center of gravity. Evidently the same proposition holds for any body or system of bodies.

20

353. Total Work Done on a Particle Expressed in Terms of Its Initial and Final Velocities and Its Mass.— Let a particle describe any path under the action of any forces. Let P be the value of the resultant of all the forces at any instant and θ the angle between P and the tangent to the path. Let m be the mass of the particle and v its velocity. The tangential component of the acceleration is dv/dt, and the tangential component of the resultant force is $P\cos\theta$. Hence there may be written the equation of motion (Art. 289)

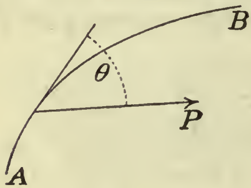


FIG. 157.

$$P\cos\theta = m(dv/dt).$$

But $dv/dt = (dv/ds)(ds/dt) = v(dv/ds)$; hence

$$P\cos\theta = mv(dv/ds), \quad \text{or} \quad (P\cos\theta)ds = mv\,dv.$$

Let s' and s'' be the values of s for any two positions A and B, v' and v'' being the corresponding values of v. Then

$$\int_{s'}^{s''}(P\cos\theta)ds = \int_{v'}^{v''} mv\,dv = \tfrac{1}{2}m(v''^2 - v'^2).$$

The first member of this equation is the total work done on the particle by all forces during the motion AB. The last member depends only upon the values of the velocity of the particle at A and at B.

Vis viva.—The quantity mv^2 is often called the *vis viva* of the particle. Using this term, the above equation expresses the proposition that the *increase in the vis viva of a particle is equal to twice the total work done upon it.* This is often called the *principle of vis viva.*

Since the development of the theory of energy, the term *vis viva* has largely given place to another, half the *vis viva* being called the "kinetic energy." The significance of this name will be explained below.

The above principle should be applied in the solution of the following examples.

EXAMPLES.

1. A body whose mass is 20 lbs. falls freely under the action of gravity. How far does it fall while its velocity changes from 40 ft.-per-sec. to 50 ft.-per-sec.? *Ans.* About 14 ft.

2. A body is projected horizontally with a velocity of 100 ft.-per-

sec. If acted upon by gravity alone, at what rate will it be moving when 20 ft. lower than the point of projection?

Ans. 106.2 ft.-per-sec.

3. A body falls vertically from a height of 5,000,000 ft. above the earth's surface. With what velocity will it reach the earth, neglecting the resistance of the air? [Use the result of Ex. 2, Art. 351.]

Ans. 16,100 ft.-per.-sec.

§ 3. *Energy of a Particle.*

354. Meaning of Work Done by a Body.— When a body moves against a resisting force, it is said to do work against that force. If the point of application of the force is displaced in a direction exactly opposite to that of the force, the quantity of work done by the body is equal to the product of the magnitude of the force into the displacement. If the displacement has some other direction, the effective displacement (or projection of the actual displacement on a line parallel to the force) must be used in computing the work.

Thus, a body thrown vertically upward does work against the force of gravity while rising, the quantity of work being equal to the product of the weight of the body into the distance it ascends. If it moves upward along an inclined plane, the effective displacement is the projection of the actual displacement upon a vertical line. A body moving through the air in any direction moves against the resisting force due to the air, and thus does work against that force. A body projected along a horizontal plane moves against the force of friction; in moving any distance the body does an amount of work equal to the product of the frictional force into the distance the body moves.

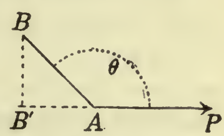


FIG. 158.

Let P be the magnitude of a constant force applied to a body at a point A (Fig. 158), and let A receive the displacement AB, making the angle θ with the direction of P. The effective displacement of A against the force is $AB' = -AB \cos \theta$. Hence the work done by the body against the force is

$$-P \times AB \cos \theta.$$

This value is positive when θ lies between 90° and 180°. The conception of work done by a body against a force is, however, to be so extended as to include negative values; although it is only when

the effective displacement and the force have opposite directions that the force can properly be said to "resist" the motion of the body. A body moving downward is thus regarded as doing negative work against gravity; in algebraic language, the force of gravity is a negative resistance.

355. Work Done by a Body Always the Negative of Work Done by a Force.— From the foregoing definitions it will be seen that whenever positive work is done by a body against a force, negative work is done by the force upon the body; and *vice versa*. The two statements are merely different ways of expressing the same fact. Whatever displacement the point of application of a force may receive, the work may be computed as done either *by the force upon the body* or *by the body against the force*. The two quantities are equal in magnitude but opposite in sign.

All the results above found in the discussion of the work done by forces upon a particle may be extended to work done by a particle against applied forces, by merely changing the sign of the work in every case.

356. Energy of a Body Defined.— When the condition of a body is such that it can do work against a force or forces that may be applied to it, the body is said to possess *energy*.

The quantity of energy possessed by a body is the amount of work it can do in passing from its present condition to some standard condition.

The unit of energy is the same as that of work.

357. Kinetic Energy of a Particle.— If a particle is in motion, it will continue to move for a time, however great a resisting force may be applied to it. It cannot be brought to rest until it has moved over some distance. By reason of being in motion, therefore, a particle can do work against any force or forces which resist the motion; that is, it possesses energy. Since this energy is possessed by the particle by reason of its motion, it is called *energy of motion* or *kinetic energy*.

A single particle can possess no energy except kinetic energy. When the definition of energy is extended to systems of particles, another form of energy will be recognized.

In defining the quantity of energy of a body (Art. 356), reference was made to a "standard condition." The full meaning of "con-

dition" cannot be explained until the theory of energy is extended to systems of particles and bodies in general. In case of a single particle, the only condition affecting its capacity to do work is its velocity; the "standard condition" is a standard velocity. So long as a particle possesses any velocity, however small, it can move against resisting forces; in estimating the energy of a particle, therefore, the standard condition will be taken as a condition of rest.

358. To Determine the Quantity of Kinetic Energy of a Particle.— It will now be shown that a particle of given mass moving with a given velocity possesses a definite quantity of kinetic energy.

In order to determine the quantity of energy possessed by a particle of mass m and velocity v, it is necessary to determine how much work will be done by the particle in coming to rest. Let it follow any path whatever and be acted upon by any forces.

By the principle proved in Art. 353, the total work done *upon* the particle while its velocity changes from v' to v'' is

$$\tfrac{1}{2}m(v''^2 - v'^2).$$

The total work done *by* the particle is the negative of this quantity, or

$$\tfrac{1}{2}m(v'^2 - v''^2).$$

In coming to rest from a velocity v the particle therefore does an amount of work $\tfrac{1}{2}mv^2$. That is,

The kinetic energy of a particle is equal to half the product of its mass into the square of its velocity.

359. Principle of Work and Energy for a Particle.— The principle expressed by the equations

$$\text{total work done upon particle} = \tfrac{1}{2}m(v''^2 - v'^2),$$
$$\text{total work done by particle} = \tfrac{1}{2}m(v'^2 - v''^2),$$

may now be called the *principle of work and energy* for a particle. Expressed in words,

During any motion of a particle its kinetic energy increases by an amount equal to the total work done upon the particle by all forces; its kinetic energy decreases by an amount equal to the total work done by the particle against all forces.

These two statements are identical in meaning, if the words increase and decrease are understood in their algebraic sense.

This principle is extremely useful in solving certain problems.

The equation of work and energy can often be written immediately, the work being expressed in terms of the forces acting and the distances through which they act, and the change of kinetic energy in terms of the mass of the particle and its initial and final velocities.

It must be remembered that a kinetic system of units (Arts. 217, 218) must be employed in writing the equation, since it is deduced from the fundamental equation of motion, $P = mp$.

Dimensions of kinetic energy.—From the formula $mv^2/2$, it is evident that kinetic energy has dimensions $\mathbf{ML}^2/\mathbf{T}^2$. This agrees with the dimensions of work when kinetic units are employed (Art. 346), as should be the case, since the equation

$$\text{work} = \text{change of kinetic energy}$$

must be homogeneous.

The following examples should be solved by applying the principle of work and energy. The value of g (the acceleration due to gravity at the surface of the earth) is taken as 32.2 foot-second units.

Examples.

1. Compute the kinetic energy of a body of 20 lbs. mass moving at the rate of 150 ft.-per-sec. *Ans.* 6,990 foot-pounds.

2. If the direction of motion of the body in Ex. 1 is vertically upward and if no force acts upon the body except its own weight, how high will it rise? (That is, how far must it move against gravity in order to do an amount of work equal to the known decrease of kinetic energy?)

3. If the resistance of the air is constantly equal to 4 lbs., how high will the body rise? *Ans.* 291 ft.

4. If a body of 10 lbs. mass is projected horizontally on a rough plane with a velocity of 50 ft.-per-sec., how far will it move before its velocity is reduced to 20 ft.-per-sec., the retarding force due to friction being constantly 5 lbs.? *Ans.* 65.2 ft.

5. A body weighing 10 lbs. falls vertically under gravity against a constant force of 1 lb. due to the resistance of the air. How far must it fall in order that its velocity may change (a) from zero to 20 ft.-per-sec.; (b) from 10 ft.-per-sec. to 20 ft.-per-sec.?

Ans. (a) 6.9 ft. (b) 5.2 ft.

6. If a body is at rest at a distance from the earth's surface equal to the radius (taken as 21,000,000 ft.), and falls under the action of its own weight, what will be its velocity on reaching the surface?

Ans. 26,000 ft.-per-sec.

7. With what velocity must a body be projected from the surface

of the earth in order that it may never return, no force except the earth's attraction being supposed to act? Does the direction of projection affect this result?

Ans. 36,700 ft.-per-sec. = 6.96 miles-per-sec.

8. A particle whose mass is 500 gr. is attached to one end of an elastic string the other end of which is fixed. The "natural length" of the string is 50 c.m.; under a pull equal to the weight of 500 gr. its length becomes 70 c.m. The string being stretched to double its natural length, the particle is held at rest and then released. If no force acts upon the particle except that due to the string, what greatest velocity will it acquire? (Take $g = 981$ C. G. S. units.)

Ans. 350 c.m.-per-sec.

9. Solve Ex. 8, taking account of the force of gravity, the string being assumed to hang vertically downward.

Ans. 210 c.m.-per-sec.

§ 4. *Energy of a System.*

360. Material System May Possess Energy Not Due to Motion.— The definition of "energy of a body" given in Art. 356 has thus far been applied only to a single particle. When the definition is applied to an actual body or to an aggregation of particles regarded as a system, our conception of energy must be enlarged. Besides the kinetic energy which may be possessed by the individual particles, the system as a whole may possess energy of another form.

In Part III will be developed the theory of the motion of a system of particles, including the general theory of energy for any material system.* We shall here give an elementary explanation of the meaning of energy of a system, depending only upon principles developed in the foregoing chapters.

361. Energy of a Deformed Elastic Body.— An example of a body which can do work even if initially at rest is furnished by an elastic string. Such a string, if not acted upon by external forces, assumes a certain "natural" length. By the application of equal and opposite forces at the ends it may be stretched, the amount of elongation depending upon the magnitude of the applied forces. If the forces are diminished, it shortens, and unless the stretching has been too great it nearly or quite resumes its original "natural" length when the forces cease to act.

---

* See Chapter XXIII.

While in the stretched condition, the string possesses energy in accordance with the definition (Art. 356); its condition is such that it can do work against applied forces. It will, in fact, do work against the forces applied to the ends if these are gradually diminished so as to permit the string to shorten.

This energy is not kinetic; it is not due to the motion of the body as a whole nor of its parts. It is due to the relations of the parts of the body to one another. Just what these relations are we do not know, since the ultimate structure of the body is unknown. It can only be said that the particles tend to assume certain relative positions, that any departure from these positions calls into action resisting forces, and that these forces will, if the external forces cease to act or decrease in magnitude, bring the particles back toward these positions.

Any elastic body, while deformed and tending to resume its "natural" shape, possesses energy irrespective of its condition of motion. Such energy may be called "energy of position," since it is due to the tendency of the particles of the body to assume definite relative positions.

362. Energy Due to Gravity.— A body near the earth whose position is such that it can move to a lower level can do work against forces resisting its descent. The condition of the body is therefore such as to satisfy the definition of energy (Art. 356). This energy is, however, possessed by the system consisting of the body and the earth rather than by the body alone. It is the weight of the body that enables it to move against the forces which resist its downward motion. The amount of work which can be done against the resisting forces is just equal to the amount of work done on the body by the earth's attraction during the descent.

This energy, like that of a deformed elastic body, may be called energy of position, since it depends upon the relative position of the bodies constituting the system and upon the action of internal forces tending to cause a definite change in their relative position.

Potential energy is the name usually given to energy of position.

After the foregoing illustrations of the meaning of potential energy, it will be well to restate the definition of energy with explicit reference to a system of bodies or of particles.

363. Energy of a System Defined.—*A system of bodies or of particles is said to possess energy when its condition is such that*

it can do work against external forces which may be applied to it.

The quantity of energy possessed by a system is the amount of work it will do against external forces in passing from its present condition to some standard condition.

It is important to notice the word "external" in this definition. The distinction between external and internal forces has been explained in Art. 119.

What shall constitute a "system" of bodies or of particles is a matter of arbitrary selection. Thus, we may regard a single body as forming a system, or we way regard the body and the earth as forming a system. The body itself possesses kinetic energy if in motion; the body and the earth regarded as a system possess energy because of their relative position.

Similarly, the particles of a deformed elastic body individually possess no energy unless they are in motion; collectively they possess energy because of the internal forces which tend to cause the body to assume a definite shape.

These cases illustrate the general principle that *potential energy is due to the action of internal forces.*

364. **Choice of "Standard Condition."**—It is now clear that the words "standard condition" in the definition of energy refer to some definite set of velocities and positions of the members of the system. The actual value of the energy depends upon what is chosen as the standard condition. This choice is governed solely by convenience. In estimating kinetic energy it is convenient to assume rest as the standard condition for each particle, since the algebraic expression for the energy is thereby simplified. In computing potential energy it is often a matter of indifference what set of positions constitutes the standard condition. In particular applications we are concerned with the difference between the values of the energy in different conditions rather than with absolute values; this difference is the same whatever be assumed as the standard condition.

365. **Value of Potential Energy Due to Gravity.**— Potential energy due to the weights of bodies near the earth is usually briefly referred to as energy possessed by the bodies themselves; although as above stated it is strictly possessed by the system of which the earth and the bodies are members. In computing its value, a hori-

zontal surface or "datum plane" must be chosen as specifying the "standard" positions of the bodies.

It may be shown that the "gravity" potential energy possessed by a particle is equal to the product of its weight into its height above the assumed reference plane; and that the total potential energy of a system of particles is the same as if the entire mass were concentrated at the center of gravity.

Let W be the weight of a particle whose height above datum is z. Its potential energy is equal to the work it can do against resisting forces in descending to the standard position,—*i. e.*, to the datum plane. If the particle is at rest in the initial position and comes to rest in the standard position, the work it has done against all forces except its weight is equal to the work done upon it by its weight (Art. 359). The value of this work is Wz; which is therefore the value of the potential energy of the particle (strictly, of the earth and the particle regarded as a system) when it is at a height z above its standard position.

Again, let a system consist of particles whose weights are W_1, W_2, . . . , at heights z_1, z_2, . . . above the plane of reference, and let $\bar{z}$ be the height of the center of gravity of the system above the same plane. Then

$$(W_1 + W_2 + \quad . \quad . \quad . \quad)\bar{z} = W_1 z_1 + W_2 z_2 + \quad . \quad . \quad . \quad .$$

The second member of this equation is the total potential energy of all the particles in the supposed positions, and the first member is the potential energy of the system on the assumption that all the particles are concentrated at the center of gravity.

Potential energy due to the weights of bodies and to their positions on the earth is one of the most important forms of energy with which we are concerned practically. In the case of streams of water, this kind of energy is available for the performance of useful work. The amount of available energy depends upon two factors,—weight of water and available fall.

366. Principle of Work and Energy for a Material System.—*During any change of condition of a material system, its total energy (kinetic and potential) increases by an amount equal to the total work done upon the members of the system by external forces; its total energy decreases by an amount equal to the total work done by its members against external forces.*

This principle is in accordance with the definition of energy of a system (Art. 363). Thus, consider the system consisting of the earth and a particle whose mass is M pounds. If the height of the particle above its standard position is z feet and if its velocity is v feet per second, the total energy of the system is

$$Mz + Mv^2/2g \text{ foot-pounds.}$$

Suppose the elevation and velocity of the particle change, their initial values being z_1, v_1, and their final values z_2, v_2. If no *external* force (*i. e.*, no force except its weight) acts upon the particle during the change, the principle of Art. 359 shows that

$$M(z_1 - z_2) = M(v_2^2 - v_1^2)/2g,$$

or

$$M(z_1 + v_1^2/2g) = M(z_2 + v_2^2/2g);$$

that is, the final value of the total energy of the system is equal to its initial value. But if the motion is resisted by an *external* force against which the particle does W foot-pounds of work, the principle of Art. 359 gives the equation

$$M(z_1 - z_2) - W = M(v_2^2 - v_1^2)/2g,$$

or

$$M(z_1 + v_1^2/2g) - W = M(z_2 + v_2^2/2g).$$

That is, the system has lost W foot-pounds of energy.

As another illustration, consider a deformed elastic body. Let its "natural" shape be taken as the standard condition for estimating its potential energy, and suppose it to be so deformed that in returning to the natural shape it can do E units of work against external forces; then (assuming its particles to possess no energy of motion) E is its total energy. Now suppose the deformation to decrease gradually until the body has done W units of work against the external forces; it can still do $E - W$ units of work in coming to the standard condition, so that its energy is now $E - W$. That is, its energy has decreased by an amount equal to the work it has done against external forces.*

---

* The reader may notice that the principle of work and energy for a material system,—in fact the very supposition that potential energy has any definite value,—involves the assumption that the total work done by the internal forces during any change of condition has a definite value which is independent of the way in which the change of condition takes place. For a rigorous discussion of this assumption, see Chapter XXIII.

367. Machine.— A machine has been defined (Art. 111) as a device for the application of force. In most cases, however, the production of motion is a part of the object of a machine. Stated completely, the function of a machine is *to do work against external forces.*

Thus, a system of pulleys used to lift a heavy body against gravity does work against the force which the body exerts upon the system by reason of its weight.

A steam-engine is designed to cause certain bodies connected with it to move against resisting forces; in accomplishing this object the moving parts of the engine do work against the forces which these bodies exert upon them.

In doing work a machine continually gives up energy (Art. 366), and it cannot continue to operate unless energy is supplied to it. The work done upon it by one set of external forces must on the whole be equal to the work done by it against another set. Thus, the external force which is doing work upon a steam-engine is the pressure of the steam upon the piston. The external forces against which the engine is doing work are (*a*) the tension in the belt which connects it with the driven machinery, and (*b*) the frictional resistances to the motion of the parts of the engine. The external work done by the engine is thus only in part utilized, that done against frictional resistances being lost.*

Efficiency.— The efficiency of a machine is the ratio of the useful work done by it to the energy supplied to it. The efficiency is always less than unity.

368. Power or Activity.— The rate at which a machine does work is called its *power* or *activity.*

It would be natural to take as the unit of power that corresponding to a unit of work done per unit time; as a foot-pound per second, or a meter-kilogram per second, or an erg per second. Such a unit is too small to be convenient for ordinary use.

The ordinary British unit is the *horse-power,* equal to 550 foot-pounds per second.

The French horse-power, or *force de cheval,* is equal to 75 meter-kilograms per second.

---

* The energy represented by the work done against friction is "lost" only in the sense of being unavailable for any useful purpose. In reality this energy is *transformed,*—principally into heat-energy. See Chapter XXIII.

The unit of activity commonly employed in electrical engineering is the *watt*, or the *kilowatt*. A watt is defined as 10^7 ergs per second, a kilowatt being therefore 10^{10} ergs per second.

Examples.

1. A stream of water is 20 ft. wide, its average depth is 3 ft., and the average velocity in the cross-section is 3 miles per hour. If there is an available fall of 200 ft., how much available potential energy is possessed by one minute's supply of water (or strictly, by the system consisting of the water and the earth)? Assume the density of water to be 62.5 lbs. per cu. ft.

Ans. One minute's supply of water is 15,840 cu. ft. or 990,000 lbs. In falling 200 ft. this water gives up 198,000,000 foot-pounds of potential energy.

2. In Ex. 1, compute the kinetic energy of one minute's supply of water. *Ans.* 297,600 foot-pounds.

3. What available H. P. is represented by the stream described in Ex. 1? *Ans.* 6,000 H. P., neglecting kinetic energy.

4. Water is to be lifted 150 ft. at the rate of 5 cu.-ft.-per-sec. What effective H. P. must be realized by the engine and pump? *Ans.* 85.2 H. P.

5. A nozzle discharges a stream 1 in. in diameter with a velocity of 80 ft.-per-sec. (*a*) How much kinetic energy is possessed by the amount of water which flows out in 1 min.? (*b*) If this energy could all be utilized by a water-wheel, what would be its power?

Ans. (*a*) 162,800 foot-pounds. (*b*) 4.93 H. P.

6. In Ex. 5, suppose the jet to drive a water-wheel connected with a pump which lifts water 20 ft. If the efficiency of the whole apparatus is 0.48, how much water is lifted per minute? *Ans.* 62.5 cu. ft.

7. A well 4 ft. square and 85 ft. deep is excavated in earth weighing 180 lbs.-per-cu.-ft. How much work is done in lifting the earth to the surface?

8. Show that the relative values of the units of activity above defined are as follows:

1 horse-power = 746 watts = 1.01385 force de cheval.

1 kilowatt = 1.34 horse-power = 1.36 force de cheval.

9. In Ex. 5, let the energy of the jet be employed in producing an electric current by means of a water-wheel driving a generator. If the efficiency of the water-wheel is 0.75 and that of the generator 0.85, what power in kilowatts is represented by the current?

§ 5. *Virtual Work.*

369. Work-Condition of Equilibrium.— Let a particle, acted upon by any number of forces, receive any displacement. The work done by the resultant of the system is equal to the algebraic sum of the quantities of work done by the several forces. (Art. 352.) If the forces are in equilibrium, their resultant is zero and the work done by it is zero. Therefore,

If a system of concurrent forces is in equilibrium, the algebraic sum of the quantities of work done by them during any displacement of their point of application is zero.

370. Virtual Displacement of a Particle.— Whatever forces may be acting upon a particle, it may at any instant be at rest, or it may be moving in any direction. Its displacement during a short interval may therefore have any direction; its actual direction being determined by previous conditions. The principle stated in the last Article is not restricted to the actual displacement, but is true for any arbitrary hypothetical displacement.

Any hypothetical displacement of a particle is called a *virtual displacement.* It may or may not coincide with the actual displacement.

371. Principle of Virtual Work.— The work done by a force during a supposed (or virtual) displacement of its point of application is called its *virtual work.* The principle of Art. 369 may therefore be stated as follows:

If a system of concurrent forces is in equilibrium, the algebraic sum of their virtual works is zero for any possible displacement.

In applying this principle, it is common to assume the displacement to be infinitesimal. This is not necessary, however, if the forces are assumed to remain constant in magnitude and direction throughout the displacement.

372. Conditions of Equilibrium.— The conditions of equilibrium for concurrent forces may be deduced from the principle of virtual work.

Let A (Fig. 159) be the point of application of several forces in equilibrium, and let it receive a displacement $AB = h$. The work done by a force of magnitude P, acting in a direction inclined at angle θ to AB, is $Ph \cos \theta$. For any number of forces, P_1, P_2,

. . . , acting at angles θ_1, θ_2, . . . with AB, the total work is

$$(P_1 \cos \theta_1 + P_2 \cos \theta_2 + \quad . \quad . \quad . \quad)h.$$

If the forces are in equilibrium, this work is zero; therefore

$$P_1 \cos \theta_1 + P_2 \cos \theta_2 + \quad . \quad . \quad . \quad = 0.$$

This is identical with the equation obtained by resolving forces parallel to AB.

By taking virtual displacements in different directions, any number of equations of the above form may be obtained, just as by resolving forces in different directions.

It is evident that, for coplanar forces, only two of these possible equations can be independent. For if the total virtual work is zero for any displacement, the only possible direction for the resultant force is perpendicular to that displacement; and if the total virtual work is zero for each of two displacements which are not parallel, the resultant must be zero.

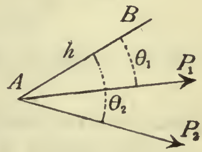


FIG. 159.

373. Forces in Three Dimensions.— The principle of virtual work is readily seen to hold for forces in three dimensions. For the proposition upon which it depends,—that the work done by the resultant of any forces is equal to the sum of the quantities of work done by the several forces,— is true without restricting the forces to a plane. This is evident from Art. 352. It may also be seen by applying the principle first to two forces and their resultant (which are coplanar), then to this resultant and a third force, and so on.

374. Application of Principle of Virtual Work.— Any problem in equilibrium of concurrent forces may be solved by the principle of virtual work. The method has no advantage over the usual method of resolving forces, so long as only a single system of con current forces is involved. It is, however, useful in the solution of many problems relating to systems of particles, and systems of rigid bodies. (See Chapter XXIII.)

In applying the principle of virtual work to a particle, any force may be eliminated from the equation of work by choosing the displacement in such direction that the virtual work of that force is zero; that is, at right angles to the direction of the force.

EXAMPLES.

1. A particle rests on a smooth inclined plane under the action of a horizontal force. Required the relation between the supporting force and the weight of the particle.

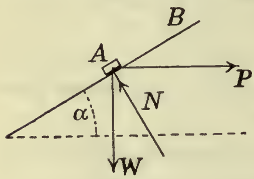


FIG. 160.

Let a = inclination of plane to horizontal, W = weight of body, P = horizontal force, N = normal reaction of plane (Fig. 160).

In order to eliminate the unknown force N, assume a displacement along the plane. Let its direction be up the plane and its length $AB = h$. The virtual displacement of P is $+h \cos a$; that of W is $-h \sin a$; and the equation of virtual work is

$$Ph \cos a - Wh \sin a = 0,$$

from which $P = W \tan a$.

The value of N may be found by taking a displacement at right angles to P.

2. A bead whose weight is W is free to slide on a smooth circular wire in a vertical plane. A string attached to the bead passes over a smooth peg at the highest point of the circle and sustains a weight P. Determine the position of equilibrium.

In Fig. 161, AB represents the vertical diameter of the circle, C the bead, O the center of the circle. Let θ = angle CAO, 2θ = angle COB. The bead is acted upon by three forces: W vertically downward, P in direction CA, normal pressure N in direction CO. Take a displacement h upward along the circle. The equation of virtual work is

$$Ph \sin \theta - Wh \sin 2\theta = 0.$$

Solving, $\cos \theta = P/2W$.

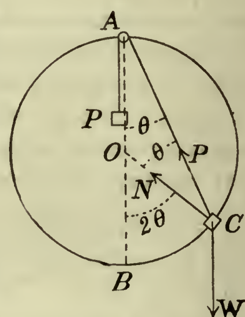


FIG. 161.

This and the preceding problem both exemplify the statement above made, that the equation of virtual work possesses no advantage over the equation obtained by resolving forces. After canceling the displacement, which is a common factor in every term of the work equation, this becomes identical with the equation obtained by resolving forces in the direction of the displacement.

PART III.

MOTION OF SYSTEMS OF PARTICLES AND OF RIGID BODIES.

CHAPTER XVIII.

MOTION OF ANY SYSTEM OF PARTICLES.

§ 1. *Motions of Individual Particles of a System.*

375. Material System Defined.— In Part II we have been concerned mainly with the motion of a single particle. An important part of the discussion has related to the problem of determining the motion of a particle when the forces applied to it are known. In the definition of force (Arts. 32 and 212) the fact was emphasized that a force is an action exerted by one particle upon another; so that the dynamical equation (force = mass × acceleration) for any particle always implies that the motion of that particle is influenced by other particles. But the attention has usually been directed particularly to a single particle.

It is often desirable to consider the motions of several particles collectively instead of individually. Any number of particles thus treated as a group may be called a *material system.*

It is found that by applying the fundamental laws of motion already explained, certain important general principles can be deduced relating to the motion of any system of particles.

The following analysis will be restricted mainly to the case in which the motion is confined to a fixed plane. It will be pointed out at the end of the chapter that most of the principles deduced are true without this restriction.

376. Coördinates of Position.— Two coördinates suffice to specify the position of any particle in a given plane. Rectangular coördinates will be employed in the following discussion.

Let the system consist of any number of particles, their masses being m_1, m_2, ... , and their coördinates of position (x_1, y_1),

(x_2, y_2), ... The motion is completely known if the coördinates of every particle are known functions of the time.

377. Motion of Mass-Center.— The *center of mass* plays an important part in the theory of the motion of a system of particles. If its coördinates are $(\bar{x}, \bar{y})$, its position is given by the equations

$$(m_1 + m_2 + \quad . \quad . \quad . \quad)\bar{x} = m_1x_1 + m_2x_2 + \quad . \quad . \quad . \quad , \qquad . \quad (1)$$

$$(m_1 + m_2 + \quad . \quad . \quad . \quad)\bar{y} = m_1y_1 + m_2y_2 + \quad . \quad . \quad . \quad , \qquad . \quad (2)$$

as in Art. 152.

If the coördinates of every particle are known functions of the time, $\bar{x}$ and $\bar{y}$ are also known functions of the time; that is, the motion of the mass-center is known from equations (1) and (2) as soon as the motion of every particle is known.

By differentiating these equations, the *velocity* of the mass-center may be determined. Its x- and y-components are $d\bar{x}/dt$, $d\bar{y}/dt$, given by the equations

$$(m_1 + m_2 + \quad . \quad . \quad . \quad)(d\bar{x}/dt) = m_1\dot{x}_1 + m_2\dot{x}_2 + \quad . \quad . \quad . \quad , \quad (3)$$

$$(m_1 + m_2 + \quad . \quad . \quad . \quad)(d\bar{y}/dt) = m_1\dot{y}_1 + m_2\dot{y}_2 + \quad . \quad . \quad . \quad . \quad (4)$$

The axial components of the *acceleration* of the mass-center are $d^2\bar{x}/dt^2$, $d^2\bar{y}/dt^2$, given by the equations

$$(m_1 + m_2 + \quad . \quad . \quad . \quad)(d^2\bar{x}/dt^2) = m_1\ddot{x}_1 + m_2\ddot{x}_2 + \quad . \quad . \quad . \quad , \quad (5)$$

$$(m_1 + m_2 + \quad . \quad . \quad . \quad)(d^2\bar{y}/dt^2) = m_1\ddot{y}_1 + m_2\ddot{y}_2 + \quad . \quad . \quad . \quad . \quad (6)$$

378. Equations of Motion for Individual Particles.— If P denotes the resultant of all the forces acting upon a particle of mass m, and p the resultant acceleration, the general equation of motion for the particle (Art. 256) is

$$P = mp.$$

Such an equation may be written for every particle of a system. Let P_1 denote the resultant of all forces acting upon the particle whose mass is m_1, and p_1 its resultant acceleration; let P_2 denote the resultant force and p_2 the resultant acceleration for the particle of mass m_2; with similar notation for every particle of the system; then there may be written the equations

$$P_1 = m_1p_1, \quad P_2 = m_2p_2, \quad . \quad . \quad . \quad . \qquad . \quad (1)$$

Each of these is a vector equation, expressing identity of direction as well as equality of magnitude.

If the motion is restricted to a plane, each of equations (1) yields two independent equations, obtained by resolving in two directions. If these directions are those of the rectangular axes of x and y, the equations may be written as follows:

$$\left.\begin{array}{ll} X_1 = m_1\ddot{x}_1, & Y_1 = m_1\ddot{y}_1; \\ X_2 = m_2\ddot{x}_2, & Y_2 = m_2\ddot{y}_2; \\ \cdots\cdots & \cdots\cdots \end{array}\right\} \quad . \quad . \quad . \quad (2)$$

In these equations X_1 and Y_1 are the axial components of P_1; $\ddot{x}_1$ and $\ddot{y}_1$ the axial components of p_1; etc.

The complete determination of the motion of every particle requires the solution of this system of simultaneous differential equations. In order that this may be possible, the forces acting upon every particle must be known functions of the coördinates and the time.

The forces acting upon any particle are in general part *external* (exerted by particles not belonging to the system) and part *internal* (exerted by other particles belonging to the system). The values of $P_1, P_2, \ldots$ and of their axial components include *all* forces, both external and internal. In many cases it is possible to express the values of these forces in terms of the coördinates. But only in exceptional cases are the resulting equations sufficiently simple to admit of complete integration. It is possible, however, by properly combining equations (2), to derive certain general equations which throw light upon the motion of the system as a whole.

379. **Elimination of Internal Forces.**—If the equations obtained by resolving along the x-axis be added, the result is

$$X_1 + X_2 + \ldots = m_1\ddot{x}_1 + m_2\ddot{x}_2 + \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Similarly, from the y-equations,

$$Y_1 + Y_2 + \ldots = m_1\ddot{y}_1 + m_2\ddot{y}_2 + \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The first members of these equations include the axial components of all forces, external and internal, acting upon every particle of the system.

By Newton's third law, the two forces which any two particles exert upon each other are equal and opposite; their components in any direction are therefore equal and opposite, and the sum of these components is zero. In the sum

$$X_1 + X_2 + \quad . \quad . \quad .$$

the x-components of the *internal* forces may therefore be omitted, their sum for the whole system being zero. The same is true of the y-components. Therefore, if X and Y are the axial components of the vector sum of the *external* forces applied to all particles of the system, equations (3) and (4) may be written

$$\left.\begin{aligned} X &= m_1\ddot{x}_1 + m_2\ddot{x}_2 + \quad . \quad . \quad . \quad ; \\ Y &= m_1\ddot{y}_1 + m_2\ddot{y}_2 + \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad (5)$$

380. Equations of Motion of Mass-Center.—Using the values of the axial components of the acceleration of the center of mass, given in Art. 377, the above equations may be written

$$\left.\begin{aligned} X &= (m_1 + m_2 + \quad . \quad . \quad . \quad)(d^2\bar{x}/dt^2); \\ Y &= (m_1 + m_2 + \quad . \quad . \quad . \quad)(d^2\bar{y}/dt^2). \end{aligned}\right\} \quad . \quad . \quad (6)$$

Equations (6) show that the acceleration of the mass-center of the system is equal to that of a particle of mass $(m_1 + m_2 + \; . \quad . \quad . \;)$ acted upon by forces whose resultant has axial components X and Y. In other words,

If a particle of mass equal to the total mass of the system were acted upon by forces equal and parallel to the external forces applied to the system, its acceleration would be equal to the actual acceleration of the mass-center of the system.

In order to determine the motion of the center of mass, it is therefore needful only to know the *external* forces.

Examples.

1. Two particles, of masses 50 lbs. and 40 lbs., are acted upon at a certain instant by parallel forces of 75 poundals and 60 poundals respectively, whose lines of action are 4 ft. apart and perpendicular to the line joining the particles. Determine (a) the position of the mass-center and (b) its acceleration at the instant named.

Ans. (b) 1.5 ft.-per-sec.-per-sec., if the forces have the same direction.

2. If the two particles of Ex. 1 attract each other with forces of 40 poundals, the remaining data being as before, compute (a) the acceleration of each particle and (b) the acceleration of the mass-center. *Ans.* (b) 1.5 ft.-per-sec.-per-sec.

3. Two bodies of masses 12 lbs. and 8 lbs. respectively, connected by an elastic string, are projected in any manner and left to

the action of gravity. At a certain instant the bodies are moving horizontally in opposite directions, each at the rate of 20 ft.-per-sec. Determine the subsequent motion of the mass-center.

Ans. Its motion will be that of a projectile having initially a horizontal velocity of 4 ft.-per-sec.

4. Take data as in Ex. 3, except that the velocity of the mass of 12 lbs. is 20 ft.-per-sec. and that of the mass of 8 lbs. is 30 ft.-per-sec. in the opposite direction. (*a*) Determine the motion of the mass-center. (*b*) If the velocities are vertical instead of horizontal, how will the solution be changed?

Ans. (*a*) Its motion will be that of a body falling from rest.

5. Show that the center of mass of any system of particles will move uniformly in a straight line if no *external* force acts upon any particle, or if the vector sum of all the external forces is zero.

§ 2. *Angular Motion of a Material System.*

381. Angular Motions of Individual Particles.—From the first pair of equations (2) (Art. 378) may be obtained the following:

$$Y_1 x_1 - X_1 y_1 = m_1(x_1 \ddot{y}_1 - y_1 \ddot{x}_1). \qquad . \qquad . \quad (1)$$

The second member of this equation may be written in the equivalent form

$$\frac{d}{dt}[m_1(x_1 \dot{y}_1 - y_1 \dot{x}_1)],$$

as may be verified by differentiation. This expression has a simple meaning. Since $m_1\dot{x}_1$ and $m_1\dot{y}_1$ are the axial components of the momentum of the particle, the moment of this momentum with respect to the origin of coördinates is $m_1(x_1\dot{y}_1 - y_1\dot{x}_1)$. Representing this by H_1, the second member of equation (1) is equal to dH_1/dt.

Since X_1 and Y_1 are the axial components of the resultant of all forces acting upon the particle, the first member of equation (1) is equal to the sum of the moments of these forces with respect to the origin of coördinates. Representing this by L_1, the equation may be written in the form

$$L_1 = dH_1/dt.$$

Let a similar equation be written for every particle; then by addition,

$$L_1 + L_2 + \quad . \quad . \quad . \quad = dH_1/dt + dH_2/dt + \quad . \quad . \quad . \quad . \quad (2)$$

382. Elimination of Internal Forces.— The first member of equation (2) is the sum of the moments of all forces, external and internal, acting upon the particles of the system. But the sum of the moments of the internal forces is zero, because the two forces acting between two particles are not only equal and opposite but are collinear, so that their moments are equal in magnitude and opposite in sign. If, therefore, the sum of the moments of the *external* forces is denoted by L,

$$L = L_1 + L_2 + \quad . \quad . \quad . \quad .$$

383. Equation of Angular Motion for the System.— Let the total moment of momentum of the system (*i. e.*, the sum of the moments of the momenta of the individual particles) be represented by H. Then

$$H = H_1 + H_2 + \quad . \quad . \quad . \quad ,$$

and equation (2) reduces to the form

$$L = dH/dt. \quad . \quad . \quad . \quad . \quad (3)$$

In deducing this equation, the origin of moments was taken at the origin of coördinates; but this may be any point in the plane of the motion.

384. Principle of Angular Momentum.— The sum of the moments of the momenta of all the particles is called the *angular momentum* of the system (Art. 327). Equation (3) expresses the proposition that

The rate of change of the angular momentum of a system about any point is equal to the sum of the moments of the external forces about that point.

The following are important special cases of this general principle:

If no external force acts upon any particle, the angular momentum of the system about every point remains constant.

If the resultant of the external forces acts always through a fixed point, the angular momentum with respect to that point remains constant.

The above equation of angular motion, together with the equations of motion of the center of mass, express all that can be determined regarding the motion of a material system from the external forces alone.

§ 3. *Effective Forces; D'Alembert's Principle.*

385. Effective Forces Defined.— The resultant of all forces (both external and internal) acting upon any particle is called the *effective force* for that particle. Such resultants for all the particles of a system make up the effective forces for the system.

For a particle of mass m and acceleration p, the effective force has the magnitude mp, its direction being that of the acceleration p. Therefore, if the position, mass and acceleration of every particle of the system are known, the effective forces are completely known. They are, in fact, forces

$$m_1p_1, m_2p_2, m_3p_3, \quad . \quad . \quad . \quad ,$$

having directions identical with those of $p_1, p_2, p_3, \quad . \quad . \quad . \quad$, and points of application coincident with the positions of the particles. The magnitude, direction and line of action of the effective force acting upon any moving particle are in general continuously changing.

386. Resultant Effective Force.— If all the effective forces for a system be combined by the methods used in combining forces in the statics of a rigid body, the resultant may be called the *resultant effective force* for the system.

For coplanar forces this resultant is a single force or a couple.

387. Relation Between Effective Forces and External Forces.— Since the effective force for any particle is the resultant of the external and internal forces acting upon that particle, the resultant effective force for the system may be found by combining all external and internal forces acting upon all particles of the system. But the resultant of all the internal forces, combined in the prescribed manner, is zero, since the entire system of internal forces is made up of stresses. It follows that

The resultant of the effective forces is equal in all respects to the resultant of the external forces.

It should be observed that the word resultant is here used in an arbitrary sense. To say that a force is the resultant of other forces means strictly that it is equivalent to them in effect. Two or more forces applied to different free particles are not equivalent to any single force. It is, however, convenient to use the term resultant whenever it is desired to express the fact that several forces are

combined according to the same rules as in the statics of a rigid body.

The meaning of the proposition stated above is that the system of effective forces and the system of external forces satisfy exactly the same conditions which are satisfied by two *equivalent systems* of forces applied to the same rigid body.

This principle leads immediately to algebraic equations of two forms: resolution equations and moment equations. Thus,

(*a*) The sum of the resolved parts of the external forces in any direction is equal to the sum of the resolved parts of the effective forces in the same direction.

(*b*) The sum of the moments of the external forces about any axis is equal to the sum of the moments of the effective forces about that axis.

If the motion is restricted to a plane, three independent equations may be written, of which at least one must be a moment equation. For motion in three dimensions six independent equations may be written.*

388. Equations of Linear and Angular Motion.—The principle that the external forces and the effective forces form equivalent systems leads immediately to the equations of motion of the mass-center given in Art. 380, and to the equation of angular motion given in Art. 383. The former are obtained by resolving along the coördinate axes, the latter by taking moments about the origin of coördinates.

The axial components of the effective force for the particle m_1 are obviously $m_1\ddot{x}_1$ and $m_1\ddot{y}_1$, and the sums of the axial components for all the particles are

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 + \ . \ . \ . = (m_1 + m_2 + \ . \ . \ . \ )(d^2\bar{x}/dt^2),$$

$$m_1\ddot{y}_1 + m_2\ddot{y}_2 + \ . \ . \ . = (m_1 + m_2 + \ . \ . \ . \ )(d^2\bar{y}/dt^2).$$

Equating these respectively to X and Y, the axial components of the resultant external force, the resulting equations are those before found for the motion of the mass-center.

Again, the moment of the effective force for the particle m_1 is

$$m_1(x_1\ddot{y}_1 - y_1\ddot{x}_1),$$

---

* These statements may be proved by reasoning similar to that employed in Arts. 104 and 173.

which, as shown in Art. 381, is equal to dH_1/dt, if H_1 is the angular momentum of m_1 about the origin; and the sum of the moments of all the effective forces is therefore dH/dt, if H is the angular momentum of the system. Equating this to L, the moment of the resultant external force, the result is the equation of angular motion given in Art. 383.

389. **System of Particles Rigidly Connected.**— In general the three independent equations, free from the internal forces, which may be written in accordance with the general principle of Art. 387, are insufficient to determine the motion completely. But if the particles are so connected that their motions are compelled to satisfy certain geometrical conditions, these conditions, together with the three dynamical equations, may suffice to determine the motion of every particle.

An important case is that in which the particles are rigidly connected, so that the distance between any two particles remains invariable. It will be seen in the following chapters that the motion of such a system is completely determined by the three dynamical equations, so that the motion of a rigid system may be determined from the external forces alone.

390. **D'Alembert's Principle.**— The principle of Art. 387, when applied to a rigid body, is known as D'Alembert's principle. It is the basis of all rules for the solution of problems relating to the motions of such bodies. The foregoing discussion makes it clear that the principle is equally true for non-rigid bodies or systems. But it is only in the case of a rigid system that the principle suffices for the complete determination of the motion.

391. **Motion in Three Dimensions.**— Although much of the foregoing discussion has referred to two-dimensional motion, most of the results may easily be extended to the case of motion in three dimensions.

The general principle of the equivalence of the external forces and the effective forces (Art. 387) is obviously unrestricted. From this follow at once the equations of motion for the mass-center, which will be three in number in the general case, and which are equivalent to the general proposition stated at the end of Art. 380. The equation of angular motion is also true for three-dimensional motion, and may be written in the same form as in Art. 383; but moments

must be taken about an axis instead of a point, and by taking moments about three axes three independent equations may be obtained. Thus, for three-dimensional motion three independent equations of resolution may be written, and also three independent moment-equations. In the case of a rigid body these six independent equations suffice to completely determine the motion.

392. **Moment of Inertia.**— In the theory of the motion of a rigid body a certain quantity called the *moment of inertia* plays an important part. The following chapter will therefore include a discussion of this quantity as a necessary preliminary to the study of the motion of a rigid body.

CHAPTER XIX.

MOMENT OF INERTIA.

§ 1. *Moment of Inertia of a Rigid Body.*

393. **Definition.**— The *moment of inertia* of a body with respect to any axis is the sum of the products obtained by multiplying every elementary mass by the square of its distance from the axis.

The value of the quantity thus defined is seen to depend upon the size and shape of the body, the distribution of its mass, and the position of the axis. For a given body the moment of inertia with respect to a line fixed in the body is a definite constant; for different axes, the moment of inertia will have different values.

394. **Moment of Inertia of a System of Discrete Particles.**— For a system of discrete particles of finite mass, the value of the moment of inertia may be computed by the formula

$$I = m_1 r_1^2 + m_2 r_2^2 + \quad . \quad . \quad . \quad = \Sigma mr^2, \qquad . \quad (1)$$

in which m_1, m_2, . . . denote the masses of the particles, r_1, r_2, . . . their respective distances from the assumed axis, and I the moment of inertia with respect to that axis.

395. **Moment of Inertia of Continuous Mass.**— In computing moments of inertia, bodies are assumed to consist of matter distributed continuously throughout space, so that any finite mass occupies a finite volume, and the moment of inertia is in general found by integration.

An *approximate value* of the moment of inertia of a continuous mass may be obtained as follows:

Let M denote the whole mass of the body, and let it be divided into small portions whose masses are ΔM_1, ΔM_2, ... Let r_1, r_2, . . . denote the distances from the assumed axis to any definite points of the elementary masses. The value of the moment of inertia is approximately

$$I = r_1^2 \Delta M_1 + r_2^2 \Delta M_2 + \quad . \quad . \quad . \quad ,$$

the approximation being closer the smaller the elements of mass are taken.

An *exact value* of the moment of inertia is found by determining the limit of the approximate value as all the elementary masses are made to approach zero, their number being increased without limit. If the variable r denotes the distance of the differential element of mass dM from the assumed axis, the exact value of the moment of inertia is

$$I = \int r^2 dM, \qquad . \quad . \quad . \quad . \quad (2)$$

the limits of the integration being so assigned as to include the entire body.

Choice of element of mass.—The differential element of mass may be of the first, second or third order; and the integration indicated in equation (2) may be either single, double or triple, depending upon the order of the differential element. The element must be so taken that all points in it have a common value of r.

396. Radius of Gyration.—The radius of gyration of a body with respect to any axis is the distance from the axis at which a single particle, of mass equal to that of the body, must be located in order that its moment of inertia may be equal to that of the body.

If M denotes the total mass of the body, I its moment of inertia with respect to any axis, and k its radius of gyration with respect to the same axis, we have

$$I = Mk^2; \quad k^2 = I/M.$$

from which k can be computed when I is known.

If the whole mass M be subdivided into parts M_1, M_2, . . . , whose radii of gyration are $k_1, k_2,$ . . . . , the moment of inertia of M is

$$I = M_1 k_1^2 + M_2 k_2^2 + \quad . \quad . \quad . \quad .$$

It may be possible to choose a differential element of M which has one or two finite dimensions, but whose radius of gyration about the given axis is known. If dM is such an element and r' its radius of gyration,

$$I = \int r'^2 dM. \qquad . \quad . \quad . \quad . \quad (3)$$

This agrees in form with equation (2), and in fact reduces to (2) when all points of dM are equally distant from the axis, since in that case $r' = r$.

397. Units Involved in Moment of Inertia.—Each term in the series whose sum is equal to the moment of inertia is the product

of a mass into the square of a length. The unit in which moment of inertia is expressed is therefore of dimensions $\mathbf{ML^2}$. In using the value of the moment of inertia in the formulas of dynamics, care must be observed that proper units of mass and length are used in computing it. The proper units to be used in any case may be seen from the nature of the particular application. In the present discussion of general principles and of methods of computing the moment of inertia, any units of mass and length may be understood.

It is usually an aid in keeping the units clearly in mind to express the moment of inertia as the product of the mass into the square of the radius of gyration. The latter quantity is obviously a simple linear magnitude.

398. **Body of Uniform Density.**— Let V denote the whole volume and M the whole mass of a body, and let ρ be the density at any point. Then

$$dM = \rho\, dV.$$

If the density is uniform throughout the body, equation (2) becomes

$$I = \rho \int r^2 dV. \qquad \qquad (4)$$

The method of applying this formula is shown in the solution of some of the following examples.

Examples.

1. Determine the moment of inertia and radius of gyration of a homogeneous right circular cylinder with respect to its axis of figure.

Let l denote the length of the cylinder, a the radius of the base, ρ the density. Taking any right section, let r, θ be the polar coördinates of any point in the cross-section, the pole of the system of coördinates being at the center. If dA is a differential element of the area of the cross-section, we may take as the element of mass a differential prism of cross-section dA and length l. The general formula for the moment of inertia becomes

$$I = \int r^2 dM = \int r^2 (\rho l\, dA) = \rho l \int r^2 dA.$$

Expressing dA in terms of the coördinates r, θ, its value is $r d\theta\, dr$; whence

$$I = \rho l \int_0^{2\pi} \int_0^a r^3 d\theta\, dr.$$

Either integration may be performed first, since the limits for each are constants. The result of the double integration between the designated limits is $I = \pi a^4 l\rho/2 = Ma^2/2$,

since the total mass M is equal to $\pi a^2 l\rho$.

If k is the radius of gyration,

$$I = Mk^2 = Ma^2/2\,; \quad k^2 = a^2/2.$$

The value of the radius of gyration is thus independent of the density.

The differential element may be so chosen that only a single integration is needed. Thus, if the element is taken as the mass included between two cylindrical surfaces of radii r and $r + dr$, $dM = 2l\rho\pi r\,dr$, and

$$I = \int r^2 dM = 2\pi l\rho \int_0^a r^3\,dr = \pi a^4 l\rho/2,$$

as before.

2. Determine the moment of inertia and radius of gyration of a homogeneous parallelopiped with respect to an axis of symmetry.

Ans. If a, b are the lengths of the edges perpendicular to the axis, $k^2 = (a^2 + b^2)/12$.

3. What is the radius of gyration of a homogeneous square prism whose dimensions in inches are $4 \times 4 \times 8$, with respect to an axis through the centroids of the square sections? How does the value of the radius of gyration depend upon the density? How upon the length? *Ans.* $k = 1.633$ ins.

4. Determine the moment of inertia and radius of gyration of a body composed of two coaxial homogeneous cylinders, the diameters of the circular sections being 4 ins. and 8 ins. respectively, and the masses 8 lbs. and 6 lbs.

Ans. $k^2 = 32/7$, $I = 64$, the inch and pound being the units of length and mass.

5. A body is made up of two portions of masses M_1 and M_2, their radii of gyration with respect to a certain axis being k_1 and k_2. Determine the moment of inertia and the radius of gyration of the whole body with respect to the same axis.

Ans. $I = M_1k_1^2 + M_2k_2^2$. $k^2 = (M_1k_1^2 + M_2k_2^2)/(M_1 + M_2)$.

6. Determine the moment of inertia and radius of gyration of a homogeneous sphere with respect to an axis through the center.

Let a be the radius of the sphere and ρ its density, and let v denote the radius of a circular section perpendicular to the axis and distant z from the center. Let dM be the mass of an element included between this section and one parallel to it at distance dz. The moment of inertia of dM is (by Ex. 1)

$$\tfrac{1}{2}v^2 dM = \tfrac{1}{2}v^2(\pi v^2\rho\,dz) = \tfrac{1}{2}\pi\rho v^4 dz,$$

and the moment of inertia of the sphere is found by integrating this differential expression between limits $z = -a$ and $z = +a$. That is,

$$I = \int_{-a}^{a} \tfrac{1}{2}\pi\rho v^4\,dz = \tfrac{1}{2}\pi\rho \int_{-a}^{a} (a^2 - z^2)^2 dz = 8\pi\rho a^5/15.$$

Since M, the mass of the sphere, is $4\pi\rho a^3/3$, the value of the moment of inertia is $2Ma^2/5$, and therefore

$$k^2 = 2a^2/5.$$

7. Determine the moment of inertia and radius of gyration of a homogeneous right circular cone with respect to its geometrical axis.

Let a be the radius of the base, h the altitude, M the mass and ρ the density. Let v be the radius of the circular section distant z from the vertex, and let dM be the elementary mass included between this section and another distant dz from it. The moment of inertia of this element is (by Ex. 1)

$$dI = \tfrac{1}{2}v^2 dM = \tfrac{1}{2}v^2(\pi v^2 dz)\rho = \tfrac{1}{2}\pi\rho v^4 dz.$$

Integrating,

$$I = \tfrac{1}{2}\pi\rho\int_0^h v^4\,dz = \tfrac{1}{2}\pi\rho\int_0^h (a^4z^4/h^4)dz = \pi a^4 h\rho/10.$$

Since $M = \pi a^2 h\rho/3$, we may write

$$I = (3a^2/10)M; \quad \therefore \quad k^2 = 3a^2/10.$$

8. Determine approximately the moment of inertia and radius of gyration of a homogeneous circular lamina of small uniform thickness with respect to an axis through the centroid parallel to the circular faces.

Taking a circular section of the lamina through the inertia-axis, let r, θ be the polar coördinates of any point in this section, the pole being at the center and the initial line coinciding with the inertia-axis. Let a be the radius of the circle, h the thickness of the lamina and ρ its density. Take as element of mass a differential prism of altitude h and cross-section $r\,d\theta\,dr$; the moment of inertia of this element is approximately $(r\sin\theta)^2(\rho hr\,d\theta\,dr)$. The required moment of inertia is therefore, approximately,

$$I = \int_0^{2\pi}\int_0^a \rho hr^3\sin^2\theta\;d\theta\,dr = \pi a^4 h\rho/4.$$

If k is the radius of gyration,

$$k^2 = I/\pi a^2 h\rho = a^2/4; \quad \therefore \quad k = a/2.$$

9. Determine the radius of gyration of a spherical shell of uniform small thickness and uniform density with respect to a diameter.

Ans. If a is the radius, $k^2 = \frac{2}{3}a^2$.

10. Determine the radius of gyration of a spherical shell of uniform density and of any thickness with respect to a diameter.

Ans. If the outer and inner radii are a and b, $k^2 = 2(a^5 - b^5)/5(a^3 - b^3)$.

399. Body of Variable Density.—If the density is not uniform throughout the body, the factor ρ cannot be placed before the sign of integration. The formula for I takes the form

$$I = \int r^2 dM = \int r^2 \rho dV. \quad . \quad . \quad . \quad (5)$$

The integration cannot be effected unless the law of variation of ρ throughout the body is known.

EXAMPLES.

1. Compute the radius of gyration of a right circular cylinder with respect to its axis of figure, assuming the density to be given by the equation $\rho = \rho_0 + cr$, r being the distance from the axis and c a constant.

Proceeding as in Ex. 1, Art. 398, we may write at once

$$I = l\int_0^{2\pi}\int_0^a \rho r^3\, d\theta\, dr = l\int_0^{2\pi}\int_0^a (\rho_0 + cr)r^3\, d\theta\, dr.$$

Integrating and reducing,

$$I = \pi l(5\rho_0 a^4 + 4ca^5)/10 = \pi a^4 l(4\rho_1 + \rho_0)/10,$$

ρ_1 being the density at the outer surface.

The value of M may be found by integration:

$$M = \pi a^2 l(2\rho_1 + \rho_0)/3.$$

Hence $\qquad k^2 = I/M = [(4\rho_1 + \rho_0)/(2\rho_1 + \rho_0)](3a^2/10).$

If $\rho_0 = \rho_1 = \rho$, these values reduce to those found for the case of uniform density.

2. Determine the moment of inertia of a sphere with respect to a diameter, assuming the density to vary directly as the distance from the center, being zero at the center and ρ_1 at the surface.

Ans. $M = \pi a^3 \rho_1$; $\quad k^2 = 4a^2/9$.

400. Relation Between Moments of Inertia With Respect to Parallel Axes.— The moment of inertia of a body with respect to any axis is equal to its moment of inertia with respect to a parallel axis through the center of mass plus the product of the whole mass into the square of the distance between the two axes.

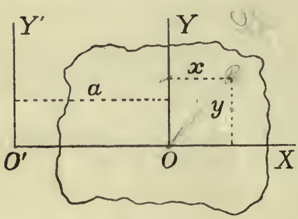


FIG. 162.

Let Fig. 162 represent a section of the body by a plane perpendicular to the assumed axes, the point O representing the central axis and O' an axis distant a from the central axis. Let $O'OX$ be the trace of a plane containing both axes; OY and $O'Y'$ the traces of planes perpendicular to $O'OX$. Let x, y be the perpendicular distances of any point of the body from the planes OY

and OX respectively, the distances of the same point from $O'Y'$ and $O'X$ being therefore $x + a$ and y.

Let I denote the moment of inertia of the body with respect to the axis O, and I' the moment of inertia with respect to the axis O'. Then

$$I = \int(x^2 + y^2)dM;$$

$$I' = \int[(x + a)^2 + y^2]dM = \int(x^2 + y^2 + a^2 + 2ax)dM$$
$$= \int(x^2 + y^2)dM + a^2\int dM + 2a\int x\,dM,$$

the limits of integration in every case being so taken as to include the whole body.

In the final expression for I', the first term is equal to I and the second to a^2M. The third term is equal to zero, since the plane OY (from which x is measured) contains the center of mass. (See Art. 159.) Hence

$$I' = I + Ma^2, \quad . \quad . \quad . \quad . \quad (1)$$

and the proposition is proved.

Relation between radii of gyration.—If k and k' are the radii of gyration of the body with respect to the axes O and O' respectively, $I = Mk^2$, $I' = Mk'^2$, and therefore

$$k'^2 = k^2 + a^2. \quad . \quad . \quad . \quad . \quad (2)$$

Examples.

1. Determine the moment of inertia and radius of gyration of a homogeneous right circular cylinder with respect to an axis coinciding with one of the straight elements of the surface.

Ans. If a = radius of circular section, $k^2 = 3a^2/2$.

2. Determine the moment of inertia and radius of gyration of a homogeneous square prism with respect to an axis coinciding with one of the edges. Apply the result to a prism whose mass is 12 lbs. and whose linear dimensions in inches are $4 \times 4 \times 8$.

Ans. If the axis coincides with one of the longer edges, $k = (8\sqrt{6})/3$.

3. Determine the moment of inertia and radius of gyration of a homogeneous right circular cone whose altitude is 6 ft. and the radius of whose base is 2 ft., with respect to an axis parallel to the geometrical axis and 2 ft. from it; the whole mass being M lbs

Ans. $k^2 = 5.2$ ft.^{2}

4. Determine the moment of inertia and radius of gyration of a homogeneous right circular cylinder with respect to an axis through the center of mass coinciding with the diameter of a right section.

Let l be the length of the cylinder, a the radius of the base, and ρ the density. Let dM be the mass included between two circular sections distant x and $x + dx$ from the center of mass. The radius of gyration of dM with respect to a diameter is $a/2$; with respect to a parallel axis through the centroid of the cylinder the square of its radius of gyration is therefore $x^2 + a^2/4$, and its moment of inertia is

$$dI = (x^2 + a^2/4)dM = (x^2 + a^2/4)(\pi a^2 \rho dx).$$

Integrating between limits $x = -l/2$ and $x = l/2$,

$$I = \pi a^2 l \rho (3a^2 + l^2)/12;$$
$$k^2 = (3a^2 + l^2)/12.$$

5. Determine the moment of inertia and radius of gyration of a homogeneous right circular cylinder with respect to an axis coinciding with a diameter of the base. *Ans.* $k^2 = (3a^2 + 4l^2)/12$.

6. Determine the moment of inertia and radius of gyration of a homogeneous right circular cone with respect to an axis through the vertex perpendicular to the geometrical axis.

Ans. If h is the altitude and a the radius of the base, $k^2 = 3(h^2 + a^2/4)/5$.

401. Product of Inertia.—The sum of the products obtained by multiplying every elementary mass of a body by the product of its distances from two planes perpendicular to each other is called the *product of inertia* of the body with respect to those planes.

For a system of discrete particles, let $m_1, m_2, \ldots$ be the masses of the particles, $x_1, x_2 \ldots$ their distances from one plane and $y_1, y_2, \ldots$ their distances from the other; and let H denote the product of inertia. Then

$$H = m_1x_1y_1 + m_2x_2y_2 + \ldots = \Sigma mxy. \quad (1)$$

For a continuous mass, let x and y denote the distances of an element of mass dM from the two planes; then

$$H = \int xy\, dM. \quad (2)$$

EXAMPLES.

1. Compute the product of inertia of a homogeneous rectangular parallelopiped with respect to planes containing two intersecting faces.

The three linear dimensions of the body being a, b and c, let the product of inertia be found with respect to two intersecting faces parallel to the dimension c. The traces of these planes are represented by OX and OY (Fig. 163). The density being ρ, the element of mass dM may be taken equal to $\rho c\, dx\, dy$, and

$$H = \int_0^a \int_0^b \rho cxy\, dx\, dy = \rho a^2 b^2 c/4 = Mab/4.$$

2. Compute the product of inertia of a homogeneous right circular cylinder with respect to a plane containing one of the bases and a plane tangent to the cylindrical surface.

Ans. If a = radius of base and l = altitude, $H = Mal/2$.

3. Compute the product of inertia of a homogeneous right circular cylinder with respect to a plane containing the geometrical axis and any plane perpendicular to that axis. *Ans.* $H = 0$.

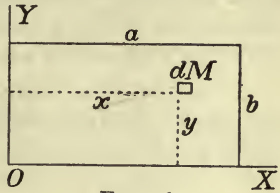


FIG. 163.

402. Product of Inertia of a Body Having a Plane of Symmetry.— The product of inertia of a body of uniform density with respect to two planes, one of which is a plane of symmetry, is zero. For, let x be the distance of any point from the plane of symmetry and y its distance from the other plane. Corresponding to any element of mass for which $x = a$, $y = b$, there is an equal element for which $x = -a$, $y = b$. The sum of the products of inertia of such a pair of elements is zero, hence the product of inertia of the whole mass is zero.

§ 2. *Moment of Inertia of a Plane Area.*

403. Definition.— In the theory of the strength and elasticity of beams, columns and shafts, there is involved a quantity whose value is given by an expression analogous in form to the expression for the moment of inertia of a solid body. The importance of this quantity, and its analogy to the moment of inertia as above defined, make it desirable to give it attention here. It may be defined as follows:

The moment of inertia of a plane area with respect to any axis is the sum of the products obtained by multiplying every elementary area by the square of its distance from the axis.

In the most important applications the axis is taken either in the plane of the given area or perpendicular to it. The moment of inertia of a plane area with respect to an axis perpendicular to its plane is called a *polar* moment of inertia.

The term radius of gyration is also used in connection with areas, being defined by the equation

$$Mk^2 = I,$$

in which I is the moment of inertia of the area, k its radius of gyration and M the total area.

It is obvious from the definition that the moment of inertia of a plane area may be computed by a method similar in every respect to that used in case of a mass, except that elements of area must be used instead of elements of mass.

It may be noticed also that the proposition of Art. 400 regarding the relation between moments of inertia with respect to parallel axes holds, using total area instead of total mass.

Examples.

1. Determine the moment of inertia and radius of gyration of a circular area with respect to an axis through the center perpendicular to the plane.

[Follow the method employed in solving Ex. 1, Art. 398, but use the element of area $r\,d\theta\,dr$ instead of the element of mass $\rho l r\,d\theta\,dr$. The radius of gyration of a circular section of the cylinder has the same value as that of the cylinder.]

2. Show that the radius of gyration of a circular area with respect to a diameter is half the radius of the circle.

3. If b and h are the sides of a rectangle, prove that its moment of inertia with respect to an axis containing the centroid and parallel to the side b is $bh^3/12$. From this result and the proposition of Art. 400 deduce the value of the moment of inertia with respect to an axis coinciding with a side of the rectangle. *Ans.* $k^2 = h^2/3$.

4. Compute the moment of inertia of a triangular area of base b and altitude h with respect to an axis containing the vertex and parallel to the base. *Ans.* $I = bh^3/4$; $k^2 = h^2/2$.

5. From the result of Ex. 4 and the proposition of Art. 400, determine the moment of inertia and radius of gyration of a triangular area with respect to an axis parallel to the base and containing the centroid; also with respect to the base.

Ans. With respect to central axis, $I = bh^3/36$, $k^2 = h^2/18$. With respect to base, $I = bh^3/12$, $k^2 = h^2/6$.

6. Prove that the radius of gyration of an elliptic area of semi-axes a and b, with respect to the diameter whose length is $2a$, is $b/2$.

404. Product of Inertia of Plane Area.— The sum of the products obtained by multiplying every element of a plane area by the product of its distances from a pair of rectangular planes is called the *product of inertia* of the whole area with respect to those planes.

The most important case is that in which the two planes are perpendicular to the plane of the given area. In this case the attention

may be confined to the lines in which the two assumed planes intersect the plane of the given area, and the definition may be stated as follows:

The product of inertia of a plane area with respect to a pair of rectangular axes lying in its plane is the sum of the products obtained by multiplying every element of area by the product of its distances from the two axes.

Reasoning as in Art. 402, it is seen that if one of the axes is an axis of symmetry of the given area, the product of inertia is equal to zero. It will be shown presently that, for any area whatever, there is a pair of axes through every point with respect to which the product of inertia is zero.

405. Products of Inertia With Respect to Different Pairs of Axes Intersecting in the Same Point.— Let OX and OY (Fig. 164) be any pair of rectangular axes lying in the plane of a given area, and let B represent the moment of inertia with respect to OX, A the moment of inertia with respect to OY, and H the product of inertia with respect to OX and OY. Take a second pair of rectangular axes OX', OY', the angles XOX' and YOY' being each equal to θ. Let H' denote the product of inertia with respect to this new pair of axes.

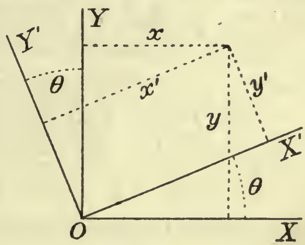


FIG. 164.

If x, y are the coördinates of any point referred to OX and OY, and x', y' its coördinates referred to OX' and OY',

$$x' = x \cos\theta + y \sin\theta, \quad y' = -x \sin\theta + y \cos\theta;$$

$$\begin{aligned} x'y' &= -x^2 \sin\theta\cos\theta + xy(\cos^2\theta - \sin^2\theta) + y^2 \sin\theta\cos\theta \\ &= xy\cos 2\theta + \tfrac{1}{2}(y^2 - x^2)\sin 2\theta. \end{aligned}$$

Hence
$$\begin{aligned} H' &= \int x'y'dM \\ &= \cos 2\theta \cdot \int xy\,dM + \tfrac{1}{2}\sin 2\theta \cdot \left[\int y^2 dM - \int x^2 dM\right] \\ &= H\cos 2\theta + \tfrac{1}{2}(B - A)\sin 2\theta. \end{aligned}$$

If the moments of inertia and the product of inertia with respect to OX and OY are known, H' can thus be computed for any value of θ.

406. Product of Inertia Zero for Certain Axes.— The axes OX' and OY' can be so chosen that H' is zero. For, equating the above value of H' to zero gives

$$\tan 2\theta = 2H/(A - B).$$

This equation gives a real value of θ for any real values of A, B and H; a pair of axes can therefore always be found for which the product of inertia is zero. Moreover, there is but one pair of axes satisfying this condition: for any two values of 2θ having the same tangent differ by some multiple of 180°, so that any two values of θ satisfying the above equation differ by some multiple of 90°.

407. Moments of Inertia With Respect to Different Axes Through the Same Point.— The moment of inertia of a plane area with respect to each of two rectangular axes through a given point, and the product of inertia with respect to the same axes, being known, the moment of inertia with respect to any other axis through that point may be determined.

Thus, referring to Fig. 164, let I be the moment of inertia with respect to OX'. Its value is

$$\begin{aligned} I &= \int y'^2 dM = \int(-x \sin\theta + y\cos\theta)^2 dM \\ &= \sin^2\theta \cdot \int x^2 dM + \cos^2\theta \cdot \int y^2 dM - 2\sin\theta\cos\theta \cdot \int xy\, dM \\ &= A\sin^2\theta + B\cos^2\theta - 2H\sin\theta\cos\theta. \qquad (1) \end{aligned}$$

If OX and OY are the pair of axes for which the product of inertia is zero,

$$I = A\sin^2\theta + B\cos^2\theta. \qquad (2)$$

EXAMPLES.

1. Find the moment of inertia and radius of gyration of a rectangular area of sides a and b with respect to a central axis inclined 30° to the side a. *Ans.* $k^2 = (a^2 + 3b^2)/48$.

2. Prove that the moment of inertia of a square area has the same value for all axes through the centroid.

3. Determine the moment of inertia and radius of gyration of a square, the length of whose side is 4 ins., with respect to an axis through the intersection of two sides and inclined at an angle of 60° to one side.

[Take the axes of x and y coincident with two sides of the square and compute A, B, H. Since H is not zero, equation (1) must be used.]

4. Show that the sum of the moments of inertia of a plane area with respect to two rectangular axes lying in its plane is equal to the moment of inertia with respect to an axis perpendicular to the plane of the area and containing the point of intersection of the rectangular axes. It follows that this sum has the same value for every pair of rectangular axes drawn through the same point.

408. Principal Axes and Principal Moments of Inertia.— If the values of the moment of inertia for all axes through a given point be compared, the greatest and least values are found for the axes with reference to which the product of inertia is zero.

Thus, the value of I given by equation (2) of Art. 407 may be expressed in the form

$$I = (A - B)\sin^2\theta + B. \qquad (3)$$

Suppose A greater than B. As θ increases from 0 to 90°, the value of I increases from B to A. As θ increases from 90° to 180°, the value of I decreases from A to B. Hence A is the greatest and B the least value of I.

The maximum and minimum values of the moment of inertia for axes through a given point are called *principal moments of inertia* for that point. The corresponding axes are called *principal axes* for that point.

If the two principal moments of inertia for axes passing through a given point are equal, the moments of inertia for all axes through that point are equal. This is obvious from equation (2).

409. Axis of Symmetry a Principal Axis at Every Point. — If an area possesses an axis of symmetry, it is a principal axis at every point. That is, the axis of symmetry and any line perpendicular to it in the plane of the area are principal axes at their point of intersection. For the product of inertia for such a pair of axes is zero, as may be proved by the method used in Art. 402.

EXAMPLES.

1. Determine the principal moments of inertia of a rectangle of sides 4 ins. and 6 ins. for axes through the middle point of the longer side.

2. Determine the principal moments of inertia of an isosceles triangle with respect to axes through the centroid.

3. Determine the principal moments of inertia of an area consisting of a square and an isosceles triangle whose base coincides with

a side of the square, with respect to axes passing through the vertex of the triangle.

4. Determine the principal moments of inertia of the area described in Ex. 3 with respect to central axes.

5. For what axis, passing through the intersection of two sides of a rectangle, is the moment of inertia a maximum?

Taking axes of x and y coinciding with the two sides of the rectangle, determine the values of A, B and H to be used in the formula of Art. 406 for determining θ. The result is $\tan 2\theta = 3ab/2(a^2 - b^2)$.

6. Determine the greatest and least moments of inertia of a rectangle with respect to axes passing through the intersection of two sides.

410. Radius of Gyration With Respect to Any Axis Through a Given Point.— Let OX and OY (Fig. 165) be principal axes of a given plane area, A being the moment of inertia with respect to OY and B the moment of inertia with respect to OX, a and b the corresponding radii of gyration, and M the total area, so that $A = Ma^2$, $B = Mb^2$. Let I be the moment of inertia and k the radius of gyration with respect to an axis OP, inclined to OX at an angle θ. Then (Art. 407)

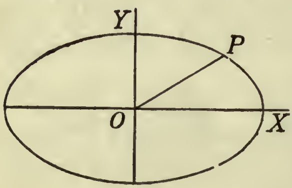


FIG. 165.

$$I = A \sin^2\theta + B \cos^2\theta;$$

$$\therefore \quad k^2 = a^2 \sin^2\theta + b^2 \cos^2\theta. \quad . \quad . \quad . \quad (4)$$

411. Inertia-Ellipse.— If in Fig. 165 the point P be so taken that the distance OP depends in some assumed manner upon the value of k, the point P will describe a curve as θ varies. If this curve is known, the value of k for any axis can be determined. The most convenient representation results from the assumption that OP is inversely proportional to k.

Let the distance OP be represented by r, and assume

$$r = ab/k,$$

r being thus a linear magnitude. From equation (4),

$$\frac{a^2b^2}{r^2} = a^2 \sin^2\theta + b^2 \cos^2\theta, \quad . \quad . \quad . \quad (5)$$

which is the polar equation of the curve described by the point P. If x and y are the rectangular coördinates of P,

$$r \sin \theta = y; \quad r \cos \theta = x;$$

and the rectangular equation of the curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which represents an ellipse of semi-axes a and b.

The most convenient method of using the ellipse depends upon the following property:

If a tangent be drawn to the ellipse, inclined at angle θ to OX, the perpendicular distance from the origin to this tangent (represented by p) is given by the equation *

$$p^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta.$$

Therefore, comparing with equations (4) and (5),

$$p = ab/r = k.$$

That is, the radius of gyration with respect to any axis through O is equal to the perpendicular distance between this axis and the parallel tangent to the ellipse.

This ellipse is called the *ellipse of inertia* for the point O. It is seen that its principal diameters lie in the principal axes of inertia (Art. 408); the principal semi-axis lying in OX is the radius of gyration with respect to OY; and the principal semi-axis lying in OY is the radius of gyration with respect to OX.

Central ellipse.— For every point in the plane of the area, there is a definite inertia-ellipse whose center is at that point. The ellipse whose center is at the centroid of the area is called the *central ellipse.*

EXAMPLES.

1. Determine the central ellipse for a rectangular area.

2. Show that the central ellipse for the area of an ellipse is a similar ellipse. What are its semi-axes? *Ans.* $a/2$ and $b/2$.

3. Determine the central ellipse for an isosceles triangle.

4. Show that the central ellipse for an equilateral triangle is a circle, and determine its radius.

Ans. $(a\sqrt{6})/12$, if a is the side of the triangle.

5. Determine the central ellipse for a regular hexagon.

Ans. A circle of radius equal to $a\sqrt{5/24}$.

* Smith's "Conic Sections," Art. 115.

CHAPTER XX.

MOTION OF A RIGID BODY: TRANSLATION; ROTATION ABOUT A FIXED AXIS.

§ 1. *Simple Motions of a Rigid Body.*

412. **Coördinates of Position.**— Any quantities whose values serve to specify the position of every particle of a body are called its *coördinates of position.*

It may be shown that six coördinates are sufficient to specify the position of a rigid body which is free to move in three dimensions of space. These six quantities may be chosen in various ways.

To show that six coördinates are sufficient, notice that if three points in the body, not lying in a right line, are fixed, every point is fixed. Let A, B, C be three such points. The position of any one, as A, is given by its three rectangular coördinates x_1, y_1, z_1 with respect to an assumed set of fixed axes. The position of B is then determined by the values of any two of its three rectangular coördinates x_2, y_2, z_2, since its distance from A is fixed. When the positions of A and B are given, that of C is determined by the value of one of its three rectangular coördinates, since its distances from A and B are fixed. Thus, to completely specify the positions of A, B and C (and therefore of all points of the body) it is sufficient to assign values to *six* of the nine quantities x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3.

413. **Degrees of Freedom.**— The six coördinates whose values specify the position of a rigid body may be varied independently of one another; for example, one may be varied while the other five remain constant. Such a variation causes the body to move in a particular way. This is often expressed by the statement that the body possesses one degree of freedom for every coördinate. A rigid body whose motion is unrestricted possesses six degrees of freedom.

By imposing restrictions on the motion of a body the number of degrees of freedom may be artificially reduced. The following discussion will deal mainly with motions so restricted as greatly to simplify the application of dynamical principles.

414. Motion of Translation.— If the velocities of all particles of a body are at every instant equal in magnitude and direction, the body is said to have a motion of *translation*.

In a motion of translation a particle of the body may describe any path in space. If no additional restriction is imposed, the body possesses *three degrees of freedom*. The coördinates of position of the body may, for example, be the rectangular coördinates of any one particle. The paths of all particles are alike in all respects.

415. Rotation About a Fixed Axis.— If all particles of a rigid body describe circles whose centers lie in a certain fixed line, the motion is a *rotation*. The fixed line containing the centers of the circles described by the particles is the *axis of rotation*.

If the motion is restricted to a rotation about a fixed axis, the body possesses only *one degree of freedom*. The position of the body is completely specified by the value of one coördinate, which may, for example, be the angle between two planes containing the axis of rotation, one of which is fixed in space and the other fixed in the body.

Let Fig. 166 represent a section of the body by a plane perpendicular to the axis of rotation, and let OX and OA be the traces of planes containing the axis, OX being fixed while OA turns with the body. The position of the body at any instant is completely specified by the value of the angle AOX. Call this θ. If θ is known as a function of the time the motion is completely determined.

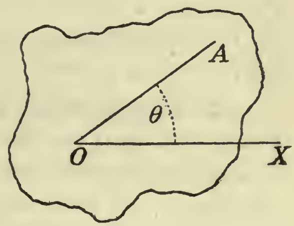


FIG. 166.

Angular motion.— All planes containing the axis of rotation or parallel to it turn through equal angles during any definite time; hence the angular motion of any one of them as OA may be called the *angular motion of the body*. This angular motion may be specified as in Art. 281.

The angle turned through in any definite time is the *angular displacement*. The angular displacement per unit time is the *angular velocity*. The increment of the angular velocity per unit time is the *angular acceleration*.

The value of the angular displacement of the body during any time Δt is

$$\Delta\theta = \theta_2 - \theta_1,$$

if θ_1 and θ_2 are the initial and final values of θ.

If ω represents the angular velocity of the body and ϕ its angular acceleration at any instant, their values may be expressed as in Art. 281:

$$\omega = d\theta/dt; \quad \phi = d\omega/dt = d^2\theta/dt^2.$$

416. Plane Motion.— Plane motion of a rigid body is a motion in which all particles move in parallel planes. This is also called *uniplanar* motion.

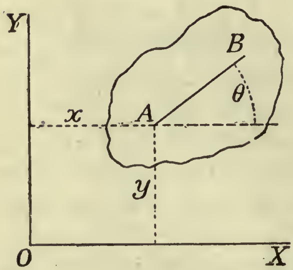


Fig. 167.

A body whose motion is thus restricted has *three degrees of freedom*. This is evident by considering that the position may be completely specified by the values of three coördinates, as follows:

Let OX, OY (Fig. 167) be a pair of fixed rectangular axes lying in a plane parallel to the motion. Let A and B be the positions of any two particles of the body lying in the plane XOY; and let x, y denote the rectangular coördinates of A and θ the angle between AB and OX. The position of the body is completely specified if values are assigned to x, y and θ.

The general case of plane motion will be considered in Chapter XXI.

417. Determination of Motion of Mass-Center.— It has been shown (Art. 380) that the motion of the center of mass of any system of particles depends only upon the external forces. To determine this motion, the entire mass of the system is conceived to be concentrated at the center of mass and to be acted upon by forces which at every instant are equal in magnitude and direction to the external forces acting upon the particles of the system.

As an example of the application of this principle, consider the motion of a projectile. If a body is thrown in any way and is then acted upon by no force except its weight, the motion of the center of mass may be determined by the results of Art. 295, which were deduced for the case of a particle. For the resultant of the external forces is constant in magnitude and direction during the motion. The same will be true of two or more bodies connected by cords or otherwise; or of any set of bodies whatever, regarded as a system.

418. Condition That Motion May Be a Translation.— If the motion of a rigid system throughout any interval is known to be a translation, the equations of motion of the mass-center suffice to determine the motion of every particle, since the paths of all particles are alike in every respect. It remains to consider under what conditions the motion will be a translation. This question may be answered by applying D'Alembert's principle (Art. 390) that *the external forces and the effective forces form equivalent systems.**

Resultant effective force.—In a motion of translation, the accelerations of all particles of the body are at every instant equal in magnitude and direction. The effective forces (each being equal to the product of the mass of a particle into its acceleration) form a system of parallel forces whose magnitudes are proportional to the masses of the particles. The resultant of this system evidently acts in a line passing through the mass-center, and its value is Mp if M is the total mass and p the acceleration.

Since the resultant external force and the resultant effective force are at every instant equal in all respects (magnitude, direction and line of action), it follows that *the line of action of the resultant of the external forces must pass through the mass-center of the system* in order that the motion may continue translatory.

The converse of this proposition is not necessarily true. But if, at any instant, all particles have equal and parallel velocities, the motion will continue to be translatory if the resultant of the external forces acts in a line containing the center of mass.

§ 2. *Rotation About a Fixed Axis.*

419. Rotation Under Any Forces.— If a body is constrained to rotate about a fixed axis, it possesses but one degree of freedom, and one dynamical equation is sufficient to determine the motion if all external forces are known. Generally, however, certain of the external forces are unknown, and additional equations are needed in order to determine them.

The method by which, practically, the motion of a body can be restricted to rotation about a fixed axis, is by means of a hinge joint (Art. 42) or something equivalent. The force or forces exerted upon

---

* The reader should bear in mind the remarks made in Art. 387 as to the meaning of this principle.

the body by the hinge are in general unknown, and are to be determined by means of the dynamical equations. D'Alembert's principle yields three independent equations for plane motion (Art. 388), and these suffice to determine the motion and the restraining forces in case of rotation about a fixed axis.* At least one of the three equations must be an equation of moments. For the other two, equations of resolution will usually be preferred.

420. Moments of Effective Forces About Axis of Rotation.— Let D'Alembert's principle be applied, taking moments about the fixed axis of rotation. The sum of the moments of the effective forces about this axis may be computed as follows:

Effective force for a particle.—Every particle of a body rotating about a fixed axis describes a circle with center in that axis. For any particle the resultant acceleration may be specified by its components along the tangent and normal to the circle. If r is the distance of the particle from the center of rotation and v its velocity, the two components are as follows (Art. 284):

Tangential component, dv/dt;

Normal component, v^2/r, directed toward the center.

If ω is the angular velocity and ϕ the angular acceleration, v is equal to $r\omega$, and since r is constant,

$$dv/dt = r(d\omega/dt) = r\phi;$$

$$v^2/r = r^2\omega^2/r = r\omega^2.$$

If the mass of the particle is m, the effective force is equivalent to the two components

$mr\phi$, directed along the tangent to the circle;

$mr\omega^2$, directed toward the center.

---

* If the applied forces tend to give the body motion parallel to the axis of rotation, the method of constraint must be such as to counterbalance this tendency, and the restraining forces will have components parallel to the axis. Again, the inertia of the particles will in general tend to cause a departure from the prescribed plane motion, which tendency must be resisted by the constraining forces. In the most general case, six dynamical equations are needed in order to completely determine the constraining forces. The following discussion is restricted to the case in which the applied forces and the distribution of mass are such that there is no tendency to depart from a condition of plane motion, so that the three dynamical equations for plane motion are sufficient to determine the constraining forces as well as the motion.

Since the moment of the latter component about the axis of rotation is zero, the *moment of the effective force* is equal to

$$mr^2\dot{\phi}.$$

Total moment of effective forces.—If the masses of the particles of the system are $m_1, m_2, \ldots$, and their distances from the axis of rotation $r_1, r_2, \ldots$, the sum of the moments of the effective forces has the value

$$(m_1r_1^2 + m_2r_2^2 + \ldots)\dot{\phi} = I\dot{\phi},$$

I being the moment of inertia of the system with respect to the axis of rotation.

421. Equation of Angular Motion.—The equation of angular motion is obtained by equating the sum of the moments of the effective forces to the sum of the moments of the external forces. Let L denote the algebraic sum of the moments of the external forces with respect to the axis of rotation; then

$$L = I\dot{\phi}.$$

If θ denotes the angle between some line in the system and a fixed reference line, the second member of the equation may be written in either of three forms, as follows:

$$L = I\dot{\phi} = I(d\omega/dt) = I(d^2\theta/dt^2). \quad . \quad . \quad (1)$$

If L is known, or can be expressed in terms of one or both of the variables θ and t, the integration of equation (1) serves to determine the motion completely, provided sufficient initial conditions are known for the determination of the constants of integration.

422. Hinge Reaction.—The rotation of a system about a fixed point is a case of constrained motion (Art. 303); a certain condition is imposed upon the motion without specification of the forces necessary to maintain that condition. In the case supposed, the constraint must be effected by something equivalent to a hinge at the center of rotation. Thus, if O (Fig. 168) is the center of rotation, any tendency of the body to depart from the stated motion is resisted by forces equivalent to a single force applied at O. This force may be called the *hinge reaction.*

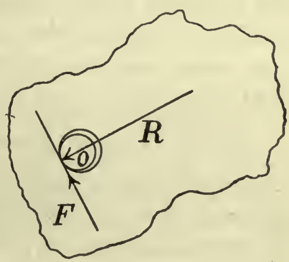


FIG. 168.

Suppose the hinge to consist of a fixed cylindrical pin fitting into a hole in the rotating body. If the surfaces were smooth at the point of contact, the pressure of the pin upon the body would act in a line passing through the axis of rotation. In reality it will depart from this direction because of the friction. Let F be the frictional force and R the normal force; then the moment of R with respect to the axis of rotation is zero, so that R does not affect the value of L. The moment of F must be included in L, its sign being always opposite to that of the existing rotation of the body. In the following discussion the friction will either be neglected or will be assumed to be included among the known external forces whose moment is L. The hinge-reaction will be regarded as acting always in a line which intersects the axis of rotation.

423. **Equations of Motion of Mass-Center.**— The value of the hinge-reaction may be determined by writing the equations of motion of the center of mass. If P is the vector sum of all external forces (including the hinge-reaction), M the mass of the body, and p the acceleration of the mass-center, these quantities satisfy the vector equation $P = Mp$. This has been proved (Art. 380) for any system of particles, whether rigid or not.

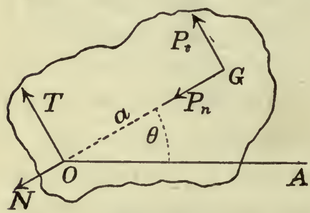


Fig. 169.

In the case of a body rotating about a fixed axis, p may be expressed in terms of the angular velocity, angular acceleration, and distance of the center of mass from the axis of rotation. Let Fig. 169 represent a section of the body by a plane perpendicular to the axis of rotation, containing the mass-center G; $OG = a =$ perpendicular from mass-center on axis of rotation; $\theta =$ angle between OG and a fixed plane OA. Then p is equivalent to the two components

$a(d^2\theta/dt^2)$ perpendicular to OG, $a(d\theta/dt)^2$ in direction GO.

Let all external forces be resolved into components respectively perpendicular and parallel to GO. Let T, N be the components of the hinge-reaction; P_t, P_n the components of *all other external forces*. The positive directions for the resolution must agree with those used in resolving p; they are shown by the arrows in Fig. 169. The figure shows P_n and P_t as if applied at G, but it is not to be

inferred that this is necessarily the fact; we are not here concerned with their lines of action.

The equations of motion of the mass-center may then take the forms

$$P_t + T = Ma(d^2\theta/dt^2); \quad . \quad . \quad . \quad (2)$$

$$P_n + N = Ma(d\theta/dt)^2. \quad . \quad . \quad . \quad (3)$$

424. Complete Solution of Problem of Rotation.—The complete solution of the problem is contained in equations (1), (2) and (3), which for convenience will be written together here:

$$L = I(d^2\theta/dt^2); \quad . \quad . \quad . \quad (1)$$

$$P_t + T = Ma(d^2\theta/dt^2); \quad . \quad . \quad . \quad (2)$$

$$P_n + N = Ma(d\theta/dt)^2. \quad . \quad . \quad . \quad (3)$$

Equation (1) does not involve the hinge-reaction, and if all other external forces are known, the motion is determined by integrating this equation. Equations (2) and (3) serve to determine T and N, the components of the hinge-reaction, after the motion is known. It will be observed that the complete integration of (1) is not necessary in order that T and N may be determined from (2) and (3).

EXAMPLES.

1. A body of mass M rotates about a fixed axis distant a from the mass-center. If no force acts upon the body except the hinge-reaction, and if its angular velocity at a certain instant is ω_1, determine (*a*) the subsequent motion and (*b*) the hinge reaction.

(*a*) In equation (1), $L = 0$, hence

$$d^2\theta/dt^2 = d\omega/dt = 0;$$

$$d\theta/dt = \omega = \text{constant} = \omega_1;$$

$$\theta = \omega_1 t + \theta_1;$$

θ_1 being the value of θ when $t = 0$. The angular velocity thus remains constant.

(*b*) The resultant acceleration of the mass-center is $a\omega^2$, directed toward the center of rotation; the resultant effective force is therefore $Ma\omega^2$ in that direction. Since the hinge-reaction R is the only external force, it must be equal in all respects to the resultant effective force.

It is evident that this result is given by equations (2) and (3). Since P_t and P_n are both zero, (2) becomes $T = 0$, and (3) reduces to $N = Ma\omega^2$.

2. A body of mass 100 lbs. rotates about a smooth hinge 2 ft. from the mass-center, being acted upon by no external force except the hinge-reaction. If at a certain instant it is rotating at the rate of 2 rev.-per-sec., what is the subsequent motion, and what is the value of the hinge-reaction?

Ans. Hinge-reaction $= N = 3,200\,\pi^2/g$ pounds-weight.

3. A body of mass 200 lbs., acted upon by gravity, rotates about a smooth horizontal axis passing through the mass-center. If at a certain instant the rate of rotation is 3 rev.-per-sec., determine the subsequent motion and the hinge-reaction.

[Notice that if the axis of rotation contains the mass-center, $a = 0$, and equations (2) and (3) simplify.]

4. A homogeneous cylinder of mass 200 lbs. and diameter 2 ft. rotates about its axis of figure (horizontal) under the action of a constant pull of 5 lbs. applied to the free end of a string which is wrapped around the cylinder. At a certain instant it is rotating at the rate of 200 rev.-per-min. *against* the pull. Determine the subsequent motion. When will the body come to rest? Determine the hinge-reaction, assuming the pull to be vertically downward.

Ans. It will come to rest in 13 sec. Hinge-reaction = 205 lbs.

5. In the preceding example, let the tension in the string be due to a suspended weight of 5 lbs., the remaining data being as before. Solve the problem completely.

[Write the equation of angular motion of the cylinder and the equation of linear motion of the suspended weight; then eliminate the tension in the string.]

Ans. It will come to rest in 13.7 sec. Hinge-reaction = 204.1 lbs.

6. A wheel-and-axle of total mass 60 lbs. is set rotating by a constant tension of 7 lbs.-force in the rope which unwinds from the axle. The radius of gyration of the body with respect to the axis of rotation is 10 ins. and the radius of the axle is 6 ins. Required (*a*) the angular velocity after 2 sec., (*b*) the angle turned through in 2 sec., and (*c*) the pressure on the hinge. (Assume no friction.)

Ans. (*a*) $0.168g$ rad.-per-sec. (*b*) $0.168g$ rad. (*c*) $67g$ poundals.

7. In the preceding example, instead of a tension of 7 lbs., assume a tension due to the weight of a suspended body of 7 lbs. mass; solve the problem completely.

8. In example 6, what weight suspended from the rope would produce a tension of 7 lbs.-force? *Ans.* 7.31 lbs.

425. Compound Pendulum.— A rigid body rotating about a horizontal axis under the action of no external force except gravity and the hinge reaction, is called a *compound pendulum*.

In Fig. 170, let the axis of rotation, perpendicular to the plane of

the figure, be projected at O, and let A be the center of mass. Let B be the lowest point reached by A, and A_0 the highest point. Let angle $AOB = \theta$, $A_0OB = \theta_0$, $AO = a$, $M =$ mass of body, $k =$ its radius of gyration with respect to the axis of rotation, $I = Mk^2 =$ moment of inertia with respect to that axis.

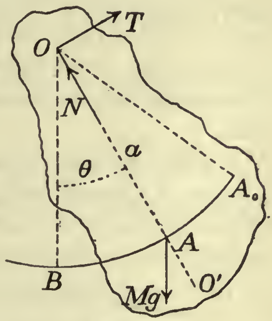


FIG. 170.

The only external force, except the hinge reaction, is the weight of the body, its value in kinetic units being Mg and its direction vertically downward. The moment of this force about the axis of rotation is

$$L = -Mga \sin \theta,$$

the negative sign being used because, in writing the equation of angular motion, the positive direction for L must agree with that for θ. Equation (1) therefore reduces to the form

$$d^2\theta/dt^2 = -(ga/k^2) \sin \theta. \quad . \quad . \quad . \quad (4)$$

The integration of this equation will determine the motion.

Equivalent simple pendulum.— The equation of motion of a simple pendulum (a particle suspended freely from a fixed point by a flexible string without weight) was found in Art. 309 to be

$$d^2\theta/dt^2 = -(g/l) \sin \theta,$$

l being the distance of the particle from the point of suspension. This is identical with the equation just derived for the compound pendulum if

$$l = k^2/a.$$

The motion of the compound pendulum is therefore the same as that of a simple pendulum of length k^2/a.

Let OA (Fig. 170) be produced to a point O' so taken that $OO' = l = k^2/a$. The point O is called the *center of suspension* and O' the *center of oscillation*. The following considerations show that as O approaches the mass-center O' recedes from it.

Let k_0 be the radius of gyration with respect to an axis through the mass-center A parallel to the axis of rotation; then

$$k^2 = k_0^2 + a^2; \quad l = k^2/a = a + k_0^2/a;$$

$$\therefore \quad (l - a)a = k_0^2, \quad \text{or} \quad OA \times O'A = k_0^2.$$

That is, the product of the distances of O and O' from the mass-center is constant.

If the center of suspension O is very near the mass-center, the value of l becomes very great; if O is very distant from the mass-center, l differs little from a.

From the above relation between OA and $O'A$ it is seen that if O' be made the center of suspension, O becomes the center of oscillation.

For the method of integrating equation (4), and of determining the time of a small oscillation, reference may be made to Art. 309. All the results there found for a simple pendulum may obviously be applied to an equivalent compound pendulum.

The first integration of equation (4) gives

$$(d\theta/dt)^2 = (2ga/k^2)(\cos\theta - \cos\theta_0). \quad . \quad . \quad (5)$$

The solution of this equation involves an elliptic integral. But for the case of a pendulum oscillating through a small angle, an approximate solution is obtained by putting θ for $\sin\theta$. With this substitution, the integration of equation (4), subject to the conditions $d\theta/dt = 0$ when $\theta = \theta_0$, $\theta = 0$ when $t = 0$, gives

$$\theta = \theta_0 \sin(t\sqrt{ga/k^2}). \quad . \quad . \quad . \quad (6)$$

EXAMPLES.

1. If the center of suspension of a homogeneous circular disc of uniform thickness is in the circumference, determine the center of oscillation (a) if the axis of suspension lies in the plane of the disc and (b) if it is perpendicular to that plane.

Ans. (a) $OO' = 5a/4$; (b) $OO' = 3a/2$; a being radius of disc.

2. The center of suspension of a homogeneous rectangular plate of sides a and b is distant r from the center of mass; required the center of oscillation, the axis being perpendicular to the plane of the rectangle. *Ans.* $(a^2 + b^2)/12r =$ distance from center of mass.

3. If, in the preceding example, $a = 40$ ins., $b = 2$ ins. and $r = 18$ ins., determine the center of oscillation and the time of a small oscillation. *Ans.* Time of oscillation $= 0.81$ sec., nearly.

4. A homogeneous straight bar, whose cross-section is very small in comparison with its length, is suspended at any point. Determine the position of the center of oscillation.

5. A homogeneous bar of small uniform cross-section, 6 ft. long, is suspended so as to rotate without retardation by friction. Determine the length of the equivalent simple pendulum when the distance

of the point of suspension from the end has each of the following values: 0, 1 ft., 2 ft., 3 ft. For what position of the point of suspension will the length of the equivalent simple pendulum be 20 ft.?

Ans. In first and third cases, $l = 4$ ft.

6. Determine the hinge-reactions in case of a compound pendulum.

Substitute in equations (2) and (3) (Art. 424) and solve for T and N. The only external force aside from the hinge-reaction is the weight of the body; its components are

$$P_t = -Mg \sin \theta, \quad P_n = -Mg \cos \theta.$$

Equation (4) gives the value of $d^2\theta/dt^2$, and (5) the value of $(d\theta/dt)^2$. Hence

$$T = Mg(1 - a^2/k^2) \sin \theta; \quad . \quad . \quad . \quad (7)$$

$$N = Mg[(2a^2/k^2)(\cos \theta - \cos \theta_0) + \cos \theta]. \quad . \quad (8)$$

7. With data as in case (a) of Ex. 1, let $\theta_0 = 90°$. Determine the magnitude and direction of the hinge-reaction when $\theta = 0$, $90°$ and $45°$ respectively.

Ans. When $\theta = 0$, $T = 0$, $N = 13Mg/5$.

8. In what position of the body has the angular velocity its greatest value? Show that if this greatest value exceeds a certain limit θ_0 is imaginary.

9. Get a first integral of equation (4), determining the constant by the condition that $d\theta/dt = \omega'$ when $\theta = 0$.

Ans. $\omega^2 = (d\theta/dt)^2 = \omega'^2 - (2ga/k^2)(1 - \cos \theta)$.

426. Resultant Effective Force.—The resultant of a system of coplanar forces is a single force or a couple. If a single force, its magnitude and direction are determined if its resolved parts in two directions are known; if, in addition, its moment about any point be known, the line of action is determined.

In case of a body rotating about a fixed axis, the resultant effective force is easily determined when the angular velocity and angular acceleration are known. If M is the mass, $I = Mk^2$ the moment of inertia with respect to the axis of rotation, p the acceleration of the mass-center, $\omega = d\theta/dt$ the angular velocity, and $\phi = d\omega/dt = d^2\theta/dt^2$ the angular acceleration, the value of the resultant effective force is as follows:

Its magnitude is Mp; its direction is that of p; its moment with respect to the axis of rotation (Art. 420) is $Mk^2\phi$, hence its line of action is at a distance $k^2\phi/p$ from the axis of rotation. This result may conveniently be put in another form, as follows:

In Fig. 171, let O represent the axis of rotation, A the mass-center, and O' the point in which the line of action of the resultant effective force Mp intersects OA. Let $OA = a$, $OO' = h$. Let Mp be replaced by two components acting at O', one acting in the direction $O'O$, the other perpendicular to that direction. The components of p in these directions are $a\omega^2$ and $a\phi$; hence the components of Mp are

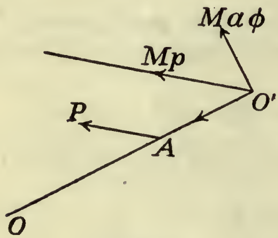


FIG. 171.

$$Ma\omega^2 \quad \text{and} \quad Ma\phi.$$

The sum of the moments of these components about O is

$$Mah\phi,$$

which must equal the moment of the resultant effective force. But (Art. 420) this is also equal to

$$Mk^2\phi;$$

hence $$ah = k^2, \quad \text{or} \quad h = k^2/a.$$

The point O' thus coincides with the point called the center of oscillation in case of a compound pendulum.

Resultant couple.—If the axis of rotation contains the center of mass, $p = 0$, hence the vector sum of the effective forces is zero. Their moment $Mk^2\phi$ is not in general zero. In this case the resultant effective force is a couple.

Resultant zero.—If the axis of rotation contains the mass-center and the angular velocity is constant, the moment of the resultant couple is zero and the resultant is therefore zero.

Motion under no external forces.—By D'Alembert's principle, the resultant effective force is zero if the resultant external force is zero. Hence if the external forces are balanced, the body can have no motion except a uniform rotation about an axis containing the center of mass. In this case the hinge-reaction is equal and opposite to the resultant of all other external forces, and is zero if this resultant is zero.

427. Unsymmetrical Body.—In the above analysis it has been assumed that the effective forces are coplanar. While it is true that, in the case of rotation about a fixed axis, the effective forces are in

parallel planes, it is not true that they are coplanar. In certain cases, however, they may be seen to be equivalent to a coplanar system. This will be true if the body is of uniform density and has a plane of symmetry perpendicular to the axis of rotation. For if the mass be regarded as made up of prismatic elements whose axes are parallel to the axis of rotation, the resultant effective force for every element will lie in the plane of symmetry. Even if there is no plane of symmetry, the mass may be so distributed with reference to a certain plane that the resultant of the effective forces lies in this plane. The conditions which must be satisfied in order that this may be true cannot here be discussed.

In the general case of an unsymmetrical body, the six dynamical equations for forces in three dimensions are needed for the complete determination of the relations among the external forces. The *motion* may, however, in all cases be determined from equation (1) of Art. 421.

CHAPTER XXI.

ANY PLANE MOTION OF A RIGID BODY.

§ 1. *Nature of Plane Motion.*

428. Coördinates of Position.— Plane motion of a rigid body has been defined in Art. 416. It was there pointed out that in the general case of plane motion three coördinates serve to specify the position of the body completely.

In analyzing plane motion it is sufficient to consider a single plane section parallel to the motion, since the position of every particle is determined if the positions of the particles in this plane section are known. In the following analysis little use is made of coördinates of position, but the geometrical relations are studied directly. When coördinates are employed, they will generally be chosen (as in Art. 416) as the rectangular coördinates of some particle A, and the angle between some fixed line and the line joining two particles. The point A may be chosen arbitrarily, but will usually be taken at the mass-center; the plane of the coördinate axes being so chosen as to contain that point.

429. Displacement.— The displacement of every particle of a rigid body (restricted to plane motion) is determined if the displacements of two particles are known. The simplest displacements of a body are *translation* and *rotation*.

The displacement is a *translation* if all particles receive equal and parallel displacements. The direction of the straight line joining any two points is left unchanged by a translation.

The displacement is a *rotation* if every particle describes the arc of a circle; the centers of the circles all lying upon a straight line called the *axis of rotation*. In the following discussion, the attention being directed to a section of the body by a plane parallel to the motion, the axis of rotation is represented by the point in which it pierces this plane. This point is called the *center of rotation;* and the terms *center* and *axis* are often used interchangeably.

The *total* displacement during any interval is called a translation or a rotation if the change from the initial to the final position can

be produced in the manner specified in the foregoing definitions, whatever the series of intermediate positions through which the particles actually pass.

430. General Plane Displacement of a Rigid Body.—*Every possible plane displacement of a rigid body is equivalent either to a rotation or to a translation.*

The displacement is completely known if the displacements of two particles are known. Let two particles initially at A and B (Fig. 172) be displaced into the positions A' and B'. Bisect AA' at C and BB' at D, and draw from C a line perpendicular to AA' and from D a line perpendicular to BB'. If these lines are not parallel, let O be their point of intersection. The triangles OAB and $OA'B'$ are equal in all their parts, since by construction $OA = OA'$, $OB = OB'$, $AB = A'B'$. Therefore the angles AOB and $A'OB'$ are equal, and the angle AOA' is equal to the angle BOB'. It is evident that the triangle OAB may be brought into coincidence with $OA'B'$ by rotation about O through an angle AOA'.

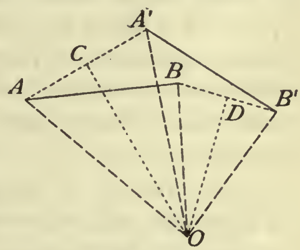


FIG. 172.

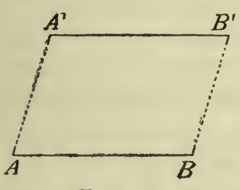


FIG. 173.

Suppose next that AA' and BB' are parallel. In this case either AA' and BB' are equal (Fig. 173) and the total displacement is a translation, or else AB and $A'B'$ are related as shown in Fig. 174, and AB may be brought to coincide with $A'B'$ by a rotation about their point of intersection O.

The proposition is thus true for all cases.

A translation may be regarded as a rotation about an infinitely distant axis. Thus, Fig. 173 represents a limiting case of Fig. 172, the lines CO and DO becoming parallel and the point O falling at infinity.

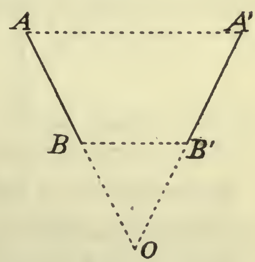


FIG. 174.

431. Instantaneous Translation and Instantaneous Rotation.—The motion may at any instant be translatory, even though it does

not continue to be so for any finite time. Again, the body may at any instant be rotating about a certain axis, even though this condition does not continue for a finite time.

Instantaneous translatory motion.— The instantaneous motion is translatory if the velocities of all particles are equal in magnitude and direction at the instant considered.

Instantaneous rotation.— The instantaneous motion is rotational if, at the instant considered, all particles in a certain axis are at rest. This axis is called the axis of instantaneous rotation, or simply the *instantaneous axis.* Every particle must be moving at right angles to the perpendicular drawn from it to the instantaneous axis, and the velocities of different particles must be proportional to their distances from the axis.

432. Nature of Instantaneous Motion in General Case.— *Any possible plane motion of a rigid body is at every instant either a translation or a rotation.*

Whatever the motion of the body, let MM' and NN' (Fig. 175) be the paths described by two particles. Let A, B be the positions of the particles at a certain instant, and A', B' their positions after a short interval. In general the *total* displacement is equivalent to a rotation about a center O, determined by the intersection of the perpendicular bisectors of AA' and BB' (Art. 430). Draw circular arcs AA', BB', with common center O. If the interval be made to approach zero, these circular arcs approach coincidence with the chords AA', BB', which themselves approach coincidence with the tangents to the paths at A and B; and the point O approaches a limiting position Q, determined by the intersection of the normals to the paths at A and B. The instantaneous motion is therefore a rotation about Q. This point is the *instantaneous center;* the line through Q perpendicular to the plane of motion is the *instantaneous axis.*

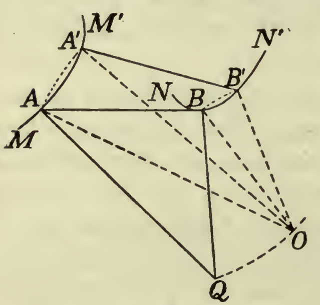


FIG. 175.

In particular cases the tangents to the paths at A and B may be parallel; if the normals are not coincident, the point Q is at infinity. In such a case the velocities of A and B are equal and parallel and

the instantaneous motion is *translatory.* Translation is thus a special case of rotation, the instantaneous axis being at infinity.

If the normals to the paths at A and B are coincident, their point of intersection is indeterminate. This case may occur if the motion is translatory and AB is perpendicular to the direction of the instantaneous motion; or if the motion is a rotation about a center lying in AB. This case may be avoided by replacing one of the points B by another C such that AC is not parallel to AB. (Compare the third case of Art. 430.)

As the body moves, the position of the instantaneous center Q (Fig. 175) in general varies continuously, both in the body and in space. The curve described by it is called a *centrode;* the curve traced in space is the *space centrode*, and the curve traced in the moving body is the *body centrode.*

It is evident from the above discussion that the position of the instantaneous center can be determined if the directions of motion of two particles are known, unless both are moving at right angles to the line joining them.

433. Angular Motion of a Body.—*Angular Displacement.*— In any plane motion of a rigid body, all lines fixed in the body and parallel to the plane of the motion turn through equal angles in any interval of time. The angle turned through by any such line is called the angular displacement of the body.

If θ is the angle between a line fixed in the body and a line fixed in space (as in Art. 416), the angular displacement is equal to $\theta'' - \theta'$, θ' being the initial and θ'' the final value of θ.

Angular velocity.— The angular velocity is the angular displacement per unit time. Representing its value by ω, it is given by the formula

$$\omega = d\theta/dt.$$

Angular acceleration.— The increment of the angular velocity per unit time is the angular acceleration. Representing it by ϕ, its value is

$$\phi = d\omega/dt = d^2\theta/dt^2.$$

These formulas for the angular motion have the same form in the general case of plane motion as in the particular case of rotation about a fixed axis (Art. 415).

434. Motions of Individual Particles.— If at any instant the angular velocity and the instantaneous axis are known, the velocity

of every particle is known. If r is the perpendicular distance of the particle from the instantaneous axis and ω the angular velocity, the particle has a velocity $r\omega$ at right angles to r.

If v_1 is the velocity of a particle A and v_2 that of a particle B, the resolved parts of v_1 and v_2 in the direction AB are equal at every instant. For otherwise the distance AB would not remain constant.

Examples.

1. The ends of a straight bar slide along intersecting lines at right angles to each other. Determine the instantaneous center at any instant.

2. If, in Ex. 1, the length of the bar is a and its angular velocity ω, determine the velocity of each end and of the middle point at the instant when the length of the bar coincides with one of the guiding lines.

3. Let the length of the bar be 12 ins. and its angular velocity constantly 500° per sec. Determine the velocity of each end and of the middle point when the bar makes an angle of 60° with one of the guiding lines.

Ans. Velocity of middle point = 4.36 ft.-per-sec., directed at angle 30° with bar.

4. One end of a bar describes a circle, the other moves along a straight line containing a diameter of the circle. Determine the instantaneous center in each of the two positions in which the bar coincides with the directing straight line.

5. In Ex. 4, let a = length of bar, r = radius of circle, v = velocity of the point describing the circle. Determine the angular velocity of the bar when it makes an angle θ with the guiding straight line. *Ans.* $\omega = (r^2 - a^2 \sin^2 \theta)^{\frac{1}{2}} v / ar \cos \theta$.

6. In Ex. 5, let A be the point describing the straight line, B the point describing the circle, C the center of the circle. Determine the velocity of the point A when ACB is a right angle. Determine the velocity of the middle point of the bar at the same instant.

7. Where is the instantaneous center, and what is the angular velocity, when the bar is in the position described in Ex. 6?

§ 2. *Composition and Resolution of Plane Motions.*

435. Component and Resultant Motions of a System of Particles.—Just as it is often useful, in the analysis of the motion of a single particle, to regard its actual velocity as made up of components, so in studying the motion of a system of particles it may be

useful to regard any actual instantaneous motion as the resultant of two or more component motions. The meaning of component and resultant motions of a system may be defined as follows:

Any definite instantaneous motion of a system of particles implies definite velocities of all the particles. Conceive two such motions M', M''; let v_1', v_2', v_3', . . . be the velocities of the individual particles in the motion M', and v_1'', v_2'', v_3'', . . . their velocities in the motion M''. Let v_1 be the vector sum of v_1' and v_1'', v_2 the vector sum of v_2' and v_2'', v_3 the vector sum of v_3' and v_3'', . . . ; and designate by M a motion of the system in which the individual particles have velocities v_1, v_2, v_3, ... Then the motion M may be regarded as the resultant of the motions M' and M''.

436. Component and Resultant Motions of a Rigid Body.— The velocities of the individual particles of a rigid body are not independent of one another. In plane motion the velocity of every particle is determined if the velocities of two particles are assigned, and even these two velocities cannot be assigned arbitrarily, but must satisfy the condition that their resolved parts along the line joining them are equal (Art. 434). Using the same notation as above, let M' and M'' be two possible instantaneous motions of a rigid body; then it may be shown that the resultant motion M is a possible motion of the rigid body.

In order to prove this, it is sufficient to show that the particles may possess velocities v_1, v_2, v_3, . . . without changing their distances apart; in other words, that the resolved parts of any two velocities v_1, v_2 along the line joining the two particles are equal. That this condition is satisfied by the velocities v_1, v_2 follows at once from the fact that it is satisfied by v_1', v_2', and also by v_1'', v_2''; and the fact that v_1 is the vector sum of v_1', v_1'', and v_2 the vector sum of v_2', v_2''. In general, therefore,

The resultant of any two instantaneous motions of a rigid body is a motion consistent with rigidity.

Any actual motion of a rigid body may therefore be regarded as the resultant of two or more simultaneous motions. It remains to consider the principles governing the composition and resolution of rigid-body motions.

437. Resultant of Two Instantaneous Motions.— It has been shown that any possible plane motion of a rigid body is at every

instant a rotation about a definite axis (reducing in a particular case to a translation). The resultant of two given motions may be determined as follows:

Let the component motions be rotations about instantaneous centers C' and C'', with angular velocities ω' and ω''; and let the resultant motion be a rotation about an instantaneous center C with angular velocity ω. It is required to determine the position of C and the value of ω. Consider separately the cases in which ω' and ω'' have (*a*) the same and (*b*) opposite angular directions.

(*a*) The resultant velocity of a particle at the instantaneous center C is zero; its two components due to the two given rotations are therefore equal and opposite. But the two component velocities are not parallel for any particle not in the line $C'C''$; hence C must lie in that line.

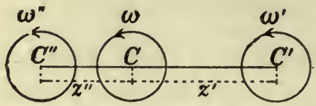


FIG. 176.

Again, if ω' and ω'' have the same angular direction (Fig. 176), it is only for a particle between C' and C'' that the two component velocities have opposite directions; C therefore lies between C' and C''.

Let z', z'' denote the distances $C'C$, $C''C$ respectively. A particle at C has a velocity $z'\omega'$ due to the rotation about C', and an opposite velocity $z''\omega''$ due to the rotation about C''; and since its resultant velocity is zero,

$$z'\omega' = z''\omega'' \quad \text{or} \quad z'/z'' = \omega''/\omega'.$$

The value of ω, the angular velocity of the resultant motion, may be found by determining the velocity of any particle not at C. The two components of velocity of a particle at C' are 0 and $(z' + z'')\omega''$; its resultant velocity is therefore $(z' + z'')\omega''$, and the angular velocity of the line CC' is

$$\omega = \omega''(z' + z'')/z';$$

or, substituting the above value of z'/z'',

$$\omega = \omega' + \omega''.$$

(*b*) In case the angular directions of ω' and ω'' are opposite, let C' and C'' (Fig. 177) represent the two instantaneous centers, and let it be assumed that ω' is greater than ω''.

As in the former case, it is seen that C must lie on the line $C'C''$, its position being such that the components of velocity of a particle at C due to the two rotations shall be equal and opposite. The components will not have opposite directions for particles between C' and C''; and since ω' is greater than ω'', the two components of velocity can be equal only if CC' is less than CC''. The point C therefore lies without the length $C'C''$ in the direction of C'. Its position must be such that

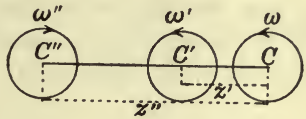


FIG. 177.

$$z'\omega' = z''\omega'', \quad \text{or} \quad z'/z'' = \omega''/\omega'.$$

To determine the angular velocity of the resultant motion, notice that the resultant velocity of C' is the sum of components 0 and $(z'' - z')\omega''$; the angular velocity of the line CC' is therefore

$$\omega = \omega''(z'' - z')/z' = \omega' - \omega''.$$

Case of two equal and opposite angular velocities.—If, in the case in which ω' and ω'' have opposite directions, their magnitudes are made to approach equality, the point C recedes from C', and in the limit, when $\omega' = \omega''$, C passes to infinity. In this case $\omega = 0$; that is, the system has no angular velocity, and the motion is a translation. The translational velocity is equal to the velocity of any particle as C', that is, it is perpendicular to $C'C''$ and has the value

$$(z'' - z')\omega'' = (C'C'')\omega''.$$

438. Analogy to Parallel Forces.—The results of the above discussion show a close analogy between the composition of rotations and the composition of parallel forces. Thus, if C' and C'' (Fig. 176 or Fig. 177) are the points of application of parallel forces of magnitudes ω' and ω'', their resultant is a force of magnitude ω applied at C; and ω is equal to $\omega' + \omega''$ or to $\omega' - \omega''$, according as the two forces have the same direction or opposite directions.

The case of two equal and opposite rotations is analogous to that of a couple; thus a translation may be regarded mathematically as a *rotation couple.*

The analogy may without difficulty be shown to hold for any number of component motions. The process of finding the angular velocity and instantaneous axis of the resultant is identical with the

process of finding the magnitude, direction and line of action of a system of parallel forces. The lines of action of the component forces coincide with the instantaneous axes of the component rotations ; the magnitude of any one of the forces is equal to the angular velocity of the corresponding motion ; and opposite rotation-directions are represented by opposite force-directions.

439. Resolution of a Rotation Into Components.— By reversing the above process, the actual motion of a body at any instant may be resolved into components.

EXAMPLES.

1. What single motion of a rigid body is equivalent to rotational velocities of 10 rad.-per-sec. and 4 rev.-per-sec. in the same direction, about instantaneous centers 5 ft. apart?

2. Two rotations of 5 rad.-per-sec. and 8 rad.-per-sec. in opposite directions, about instantaneous centers 4 ft. apart, are equivalent to what single rotation?

3. What is the resultant of two rotations of 10 rev.-per-min. in opposite directions, about centers 2 ft. apart?

4. A rotation of 4 rev.-per-sec. about a center C is equivalent to what two rotations about centers distant 1 ft. and 2 ft. from C on opposite sides?

5. A rotation of 5 rev.-per-sec. about a center C is equivalent to what two rotations about centers distant 3 ft. and 2 ft. from C in the same direction?

6. A translational velocity of 20 ft.-per-sec. is equivalent to what angular velocities about two points 8 ft. apart? What must be the direction of the line joining the proposed centers in order that the resolution may be possible?

7. Show how to resolve a given rotation into three components whose centers are given and are not collinear.

440. Translation and Rotation.— The above processes of combining or resolving rotations include the case in which one of the components is a translation. An independent treatment of this case is, however, desirable.

Composition.— Let the resultant motion of a rigid body at a certain instant be made up of two components, one of which is a rotation about the instantaneous center C (Fig. 178) with angular velocity ω, and the other a translation of velocity v in the direction CM. Since this translational velocity is the resultant velocity of C.

the instantaneous center C' of the resultant motion must lie on a line through C perpendicular to CM. If z denotes the distance CC', we must have

$$z\omega = v, \quad \text{or} \quad z = v/\omega,$$

since the resultant velocity of a particle at C' must be zero.*

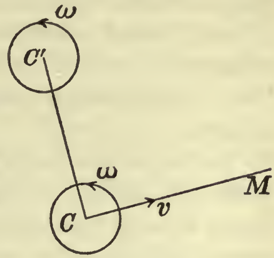


FIG. 173.

Resolution.— By reversing the above process, any given motion may be resolved into a translation and a rotation.

As a special case, any motion may be resolved into a translation and a rotation about an instantaneous center chosen at pleasure. To accomplish this, it is only necessary to select as the translational component of velocity for every particle *the velocity of the point which is to be made the instantaneous center.* This will be illustrated in treating the dynamics of plane motion of a rigid body.

EXAMPLES.

1. A rotation of 100 rev.-per-min. is to be resolved into two components, one of which shall be a translation equal to the actual velocity of a certain particle 4 ft. from the center of rotation. Determine the two components completely.

2. A rotation of 100 rev.-per-min. about a center C and a translation of 86 ft.-per-min. in a certain direction are equivalent to what resultant motion?

3. Resolve a rotation of 50 rad.-per-sec. into two components, one of which shall be a translation and the other a rotation about a center 6 ft. from the center of the given rotation.

Ans. The translational velocity is 300 ft.-per-sec. at right angles to the line joining the two centers of rotation.

441. Composition and Resolution of Accelerations.— If, in two motions M', M'' of a rigid body, a certain particle has accelerations p', p'' respectively, its acceleration in the resultant motion is the vector sum of p' and p''. This follows from Art. 436, since acceleration is *change of velocity per unit time.* A particular case of the resolution of accelerations is important in the application of D'Alembert's principle to the general case of plane motion (Art. 443).

* It will be noticed that the process of combining a rotation and a translation is exactly similar to the process of combining a force and a couple, explained in Art. 94.

If A and B are any two particles of a rigid body having any plane motion, and if the acceleration of B is resolved into two components, one of which is equal to the acceleration of A, the other component is equal to the acceleration which B would have if the body rotated about a fixed axis through A with its actual angular motion.

Let v be the velocity of A and p its acceleration; then the truth of the proposition becomes evident by considering that if, with the actual motion, there be compounded a translation with velocity $-v$ and acceleration $-p$, the particle A is reduced to rest while the angular motion is unchanged.

§ 3. *Dynamics of Plane Motion.*

442. Application of D'Alembert's Principle.— The way in which the condition of motion of a body is changing at any instant depends upon the shape and size of the body, the distribution of its mass, and the forces acting upon every particle. The influence of these several elements may be expressed by algebraic equations, derivable from the general principle known as "D'Alembert's principle."* The application of this principle to the general plane motion of a rigid body must now be considered.

443. Effective Forces.— In order to determine the effective forces, let the actual motion be resolved into two component motions in accordance with the principle stated in Art. 441, and let the effective forces for the component motions be computed separately.

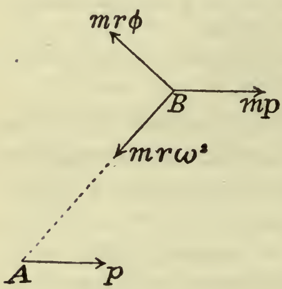


Fig. 179.

One of the component motions is a rotation about a fixed axis with the actual angular velocity and angular acceleration of the body; the other is a translation with velocity and acceleration equal to those of a particle lying in the assumed fixed axis. The position of the fixed axis may be chosen at pleasure; let it be at A (Fig. 179), and let v be the velocity and p the acceleration of a par-

---

* See Arts. 387, 390.

ticle of the body which instantaneously coincides with A. Let θ be the angular coördinate of the body (Art. 433), $\omega = d\theta/dt$ its angular velocity, and $\phi = d\omega/dt = d^2\theta/dt^2$ its angular acceleration. The two components into which the actual motion of the body is resolved are then (a) a translation with velocity v and acceleration p; (b) a rotation about a fixed axis through A with angular velocity ω and angular acceleration ϕ. Consider the effective force on any particle due to each of these component motions. Let B (Fig. 179) represent the position of a particle of mass m, and let $AB = r$.

(a) Corresponding to the translation, the effective force is equal in magnitude and direction to mp.

(b) Corresponding to the rotation, the effective force is made up of a component $mr\omega^2$ in direction BA and a component $mr\phi$ at right angles to BA.

Since the translational effective forces on different particles are proportional to their masses and have a common direction, their resultant is a force Mp applied at the mass-center, M being the whole mass of the body.

The resultant of the rotational effective forces may be determined as in Art. 426. The *moment* of this resultant about the axis of rotation is $I\phi$, I being the moment of inertia about that axis (Art. 420).

The rotational component of the motion takes place about an axis chosen at pleasure. In order to simplify the system of effective forces, let this axis contain the mass-center. Then, as shown in Art. 426, the rotational effective forces are equivalent to a couple. The moment of this couple about any axis whatever is equal to $I\phi$ if I is the moment of inertia with respect to the central axis.

The results of the discussion may be summarized as follows:

In any plane motion of a rigid body, the system of effective forces is equivalent to a force Mp applied at the mass-center and a couple of moment $I\phi$; where M is the total mass of the body, I its moment of inertia with respect to a central axis perpendicular to the plane of the motion, p the acceleration of the mass-center, and ϕ the angular acceleration of the body.

444. **Determination of Angular Motion.**—By D'Alembert's principle, the sum of the moments of the external forces about any axis is equal to the sum of the moments of the effective forces about that axis. Let the axis of moments contain the center of mass; then the moment of the resultant of the translational effective forces is

zero, and the sum of the moments of the effective forces is equal to $I\phi$, if I is the moment of inertia with respect to the central axis. Let L denote the sum of the moments of the external forces about this axis; then

$$L = I\phi.$$

This is identical with the equation of angular motion about a fixed axis containing the mass-center.

445. Independence of Translation and Rotation.— Any actual motion of a body being regarded as made up of a rotation about an axis containing the mass-center, together with a translation equal to that of the mass-center, we have now proved that these two motions take place independently. The meaning of this statement may be expressed definitely as follows:

(1) *The acceleration of the mass-center is the same as if the whole mass were concentrated at that point and acted upon by forces equal in magnitude and direction to the actual external forces.*

(2) *The angular acceleration is the same as if the mass-center were fixed and the actual external forces applied.*

446. Complete Determination of the Motion.— In the general case of plane motion, three independent dynamical equations may be written. Two of these may be the equations of motion of the mass-center, the third the equation of angular motion. In any determinate problem these three equations, together with the initial conditions for determining constants of integration, and the geometrical relations (if the motion is constrained) serve to determine the motion and the unknown forces.

Let the coördinates of the body be taken as in Art. 428. Choosing a pair of rectangular axes lying in a plane parallel to the motion and containing the mass-center, let

x, y = coördinates of mass-center;
θ = angular coördinate of body;
M = total mass of body;
$I = Mk^2$ = moment of inertia with respect to central axis;
P = vector sum of external forces;
X, Y = axial components of P;
L = sum of moments of external forces about central axis.

The three equations of motion take the form

$$M(d^2x/dt^2) = X; \quad . \quad . \quad . \quad . \quad (1)$$

$$M(d^2y/dt^2) = Y; \quad . \quad . \quad . \quad . \quad (2)$$

$$I(d^2\theta/dt^2) = L. \quad . \quad . \quad . \quad . \quad (3)$$

447. Applications.—The methods of applying the above principles will be illustrated by the solution of the following problems:

I. Determine the motion of a homogeneous circular cylinder which rolls, without sliding, on an inclined plane, under the action of gravity.

Solution.—Let β = inclination of plane to horizon; a = radius of cylinder; M = its mass; k = radius of gyration about its geometrical axis; ($k^2 = \frac{1}{2}a^2$ by Art. 398, Ex. 1;) x, y, θ = coördinates of position, the origin and axes being taken as shown in Fig. 180.

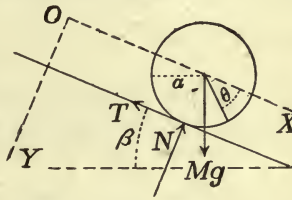


FIG. 180.

The external forces acting upon the cylinder are its weight and the reaction of the plane. The weight is equivalent to a force Mg, acting vertically downward at the mass-center. The reaction of the plane is unknown in magnitude and direction, but may be replaced by its normal and tangential components N and T, as shown in the figure. Noticing that the y-component of acceleration of the mass-center is zero, since y is constant, the three equations may be written as follows:

$$M(d^2x/dt^2) = Mg \sin \beta - T; \quad . \quad . \quad (1)$$

$$0 = Mg \cos \beta - N; \quad . \quad . \quad (2)$$

$$\tfrac{1}{2}Ma^2(d^2\theta/dt^2) = Ta. \quad . \quad . \quad . \quad . \quad (3)$$

Since the cylinder rolls without sliding, there is also the geometrical equation

$$x = a\theta + \text{constant},$$

from which $$d^2x/dt^2 = a(d^2\theta/dt^2). \quad . \quad . \quad . \quad (4)$$

These equations may be solved as follows:

Substituting in (1) the value of T from (3) and the value of d^2x/dt^2 from (4), and reducing,

$$d^2\theta/dt^2 = (2g \sin \beta)/3a.$$

Integrating and assuming that θ and $d\theta/dt$ are both zero when $t = 0$,

$$\theta = [(g \sin \beta)/3a]t^2. \quad . \quad . \quad . \quad (5)$$

This determines the motion. If the origin is so taken that $x = 0$ when $\theta = 0$,

$$x = a\theta = \tfrac{1}{3}g \sin \beta \cdot t^2. \quad . \quad . \quad . \quad (6)$$

To complete the solution the values of N and T must be found. From (1),

$$T = M(g \sin \beta - \ddot{x}) = \tfrac{1}{3}Mg \sin \beta. \quad . \quad . \quad (7)$$

From (2),

$$N = Mg \cos \beta. \quad . \quad . \quad . \quad . \quad (8)$$

II. A homogeneous circular cylinder is placed with its axis vertical on a smooth horizontal plane, and is set in motion by a constant tension applied to the free end of a flexible cord wound upon its surface. Determine the motion.

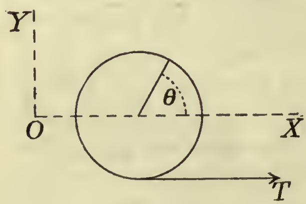


FIG. 181.

Solution. — Let M = mass of cylinder; a = its radius; k = its radius of gyration with respect to its geometrical axis; ($k^2 = \frac{1}{2}a^2$;) $I = Mk^2 = \frac{1}{2}Ma^2$; T = tension in string; x, y, θ = coördinates of position taken in the usual way, the axes being as shown in Fig. 181.

Since the only external force acting on the cylinder in the plane of the motion is T, the acceleration of the mass-center has the direction of T. The dynamical equations are

$$M(d^2x/dt^2) = T; \quad . \quad . \quad . \quad . \quad (1)$$

$$\tfrac{1}{2}Ma^2(d^2\theta/dt^2) = Ta. \quad . \quad . \quad . \quad . \quad (2)$$

If T is a known constant, the motion can be determined by the integration of these two equations. If T is unknown, the motion cannot be completely determined; but in any case the elimination of T shows that the angular acceleration and the linear acceleration bear a constant ratio to each other; that is,

$$d^2x/dt^2 = \tfrac{1}{2}a(d^2\theta/dt^2).$$

EXAMPLES.

1. A homogeneous cylinder 2 ft. in diameter rolls without sliding down a plane inclined 30° to the horizon. (*a*) Determine the dis-

tance moved by the mass-center in 2 sec., starting from rest. (*b*) If the mass of the cylinder is 40 lbs., determine the normal and tangential components of the reaction of the plane on the cylinder.

Ans. (*a*) 21.5 ft., nearly. (*b*) $T =$ one-sixth the weight of the body.

2. Solve the problem of the motion of a cylinder rolling on an inclined plane, assuming that the body has initially a motion up the plane. Apply the results to the case described in Ex. 1, assuming an initial velocity of the mass-center of 20 ft.-per-sec. up the plane.

Ans. The body will come to rest at the end of $60/g$ sec., and will then descend as in Ex. 1.

3. Determine the motion of a homogeneous sphere which rolls without sliding down a rough inclined plane, and apply the results to a sphere of 10 lbs. mass and 2 ins. diameter rolling from rest down a plane inclined 5° to the horizon.

Ans. The acceleration of the mass-center is 5/7 that of a body sliding on the plane without friction.

4. A homogeneous sphere rolls, without sliding, on the inner surface of a hemisphere. If it moves from rest under the action of no force except gravity and the reaction of the surface, determine the motion.

Let A' be the center of the hemisphere (Fig. 182), A that of the sphere, B the lowest position of A, AE that radius of the sphere which is vertical when AA' is vertical. Let $a =$ radius of sphere, $a' =$ radius of hemisphere, $\phi =$ angle $AA'B$, $\theta =$ angle between AE and vertical.

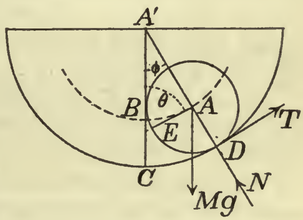


FIG. 182.

In writing the equations of motion of the mass-center, let the acceleration be resolved along the tangent and normal to the circle described by A. The components are $(a' - a)(d^2\phi/dt^2)$ and $(a' - a)(d\phi/dt)^2$. Let the pressure on the sphere at the point of contact D be resolved into a normal component N and a tangential component T, both unknown. The only other external force is the weight Mg.

The equations of motion are

$$T - Mg\sin\phi = M(a' - a)(d^2\phi/dt^2); \qquad (1)$$

$$N - Mg\cos\phi = M(a' - a)(d\phi/dt)^2; \qquad (2)$$

$$-Ta = (2Ma^2/5)(d^2\theta/dt^2). \qquad (3)$$

The angles θ and ϕ are related in a simple manner. Since the arcs ED and CD are equal, and angle $DAE = \theta + \phi$,

$$a'\phi = a(\theta + \phi). \qquad (4)$$

From equation (4),

$$a(d^2\theta/dt^2) = (a' - a)(d^2\phi/dt^2).$$

Substituting this value in (3), eliminating T between (3) and (1), and reducing,

$$\frac{d^2\phi}{dt^2} = -\frac{5g}{7(a' - a)} \sin\phi. \quad . \quad . \quad . \quad (5)$$

This equation is identical in form with that for the motion of a pendulum. The complete solution involves an elliptic integral, but one integration can be performed as in Art. 425. If the sphere is at rest when $\phi = \phi_0$,

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{10g}{7(a' - a)}(\cos\phi - \cos\phi_0). \quad . \quad . \quad (6)$$

From (1) and (5),

$$T = (2Mg \sin\phi)/7. \quad . \quad . \quad . \quad (7)$$

From (2) and (6),

$$N = Mg(17 \cos\phi - 10 \cos\phi_0)/7. \quad . \quad . \quad (8)$$

5. In Ex. 4, how great must the coefficient of friction be in order that the assumed rolling without sliding may be possible?

The coefficient of friction must be not less than the greatest value of T/N found in the above solution. Let

$$\mu = T/N = (2 \sin\phi)/(17 \cos\phi - 10 \cos\phi_0).$$

The value of μ increases with ϕ, and has its greatest value when $\phi = \phi_0$; in this position

$$\mu = (2 \tan\phi_0)/7,$$

which is the least value of the coefficient of friction compatible with the assumed motion.

6. In Ex. 4, if the sphere oscillates through a small angle about its lowest position, determine the length of an equivalent simple pendulum. *Ans.* $7(a' - a)/5$.

7. A sphere rolls, without sliding, on the outer surface of a sphere. If it starts from rest at the highest point, where will it leave the surface? (Put the normal reaction equal to zero.)

Ans. Let ϕ = angle between vertical and line joining centers; then the spheres separate when $\cos\phi = 10/17$; $\phi = 53° 58'$.

8. In Ex. 7, if the spheres are smooth, show that they will separate when $\cos\phi = 2/3$.

9. A circular cylinder whose mass-center is not in its geometrical axis rolls, without sliding, upon a horizontal plane. Let a = radius of circular section, c = distance of mass-center from geometrical axis, k = radius of gyration with respect to axis through mass-center, θ = angular displacement from position of stable equilibrium at time t. Show that

$$[g + a(d\theta/dt)^2](a^2 + c^2 + k^2 - 2ac \cos\theta)$$

remains constant during the motion.

10. A circular cylinder whose mass-center is not in its geometrical axis is placed on a smooth horizontal plane. With notation as in Ex. 9, show that

$$(k^2 + c^2 \sin^2 \theta)(d\theta/dt)^2 - 2gc \cos \theta$$

remains constant during the motion.

11. How great must the coefficient of friction be in order that a homogeneous cylinder placed on an inclined plane with axis horizontal shall roll without sliding?

448. General Method of Writing Equations of Motion.— The three equations of motion are not necessarily written as in Art. 446, although this method furnishes the simplest solution of many problems.

The equations derivable from D'Alembert's principle are of two kinds,— equations of resolution and equations of moments. All such equations are included in the two following statements:

(*a*) The sum of the resolved parts of the effective forces is equal to the sum of the resolved parts of the external forces, for any direction of resolution.

(*b*) The sum of the moments of the effective forces is equal to the sum of the moments of the external forces, for any axis.

By reasoning analagous to that used in Art. 104 in discussing the conditions of equilibrium of coplanar forces, it may be shown that only three *independent* dynamical equations can be obtained by expressing the equivalence of the effective forces and the external forces. The remarks made in Art. 387 as to the meaning of D'Alembert's principle must here be kept in mind. The statement that the system of effective forces is equivalent to the system of external forces is merely a concise expression of the fact that the two systems satisfy exactly the same conditions which are satisfied by two systems of external forces which would be equivalent in effect if applied to a rigid body. This relation of equivalence between the two systems is completely expressed by three equations falling under the above general forms (*a*) and (*b*).

The three independent equations may therefore be written in three different ways, as follows:

(1) By resolving in each of two directions and taking moments about any axis.

(2) By taking moments about each of two axes and resolving in a direction not perpendicular to the plane containing the axes.

(3) By taking moments about each of three axes not lying in one plane.*

449. Equation of Moments for Axis Not Containing Mass-Center.— The simplest moment equation, so far as regards the effective forces, is obtained by taking the axis through the mass-center; the translational effective forces are thus eliminated, since their resultant is applied at the mass-center. But it may be advantageous to take another moment axis for the purpose of eliminating certain of the external forces. In writing such an equation, the translational effective force Mp, applied at the mass-center, must not be omitted.

To illustrate the general method of writing an equation of moments, consider the problem of a cylinder rolling on an inclined plane (Art. 447, Problem I). If moments are taken about the element of contact of the cylinder and plane, the forces N and T are eliminated. The only external force whose moment is not zero is the weight of the body, and its moment is $Mga \sin \beta$. The sum of the moments of the effective forces is

$$Ma(d^2x/dt^2) + Mk^2(d^2\theta/dt^2) = (3Ma^2/2)(d^2\theta/dt^2).$$

The equation of moments is therefore

$$(3Ma^2/2)(d^2\theta/dt^2) = Mga \sin \beta,$$

which agrees with the result given in Art. 447.

Examples.

1. A uniform bar is placed in a smooth hemispherical bowl in such a position that a vertical plane through the axis of the bar contains the center of the sphere. Determine a differential equation for the motion. If a is the length of the bar and c its distance from the center of the sphere, show that its motion will be the same as that of a simple pendulum of length $c + a^2/12c$. Interpret the limiting cases $c = 0$, $a = 0$.

2. A straight bar is placed at rest with one end on a smooth horizontal plane and the other against a smooth vertical plane which is at right angles to the vertical plane containing the axis of the bar. Write three independent equations for the motion, one of which shall be independent of the unknown reactions.

---

*It is of course understood that the axes of moments are all perpendicular to the plane of the motion, and the directions of resolution parallel to that plane.

§ 4. *Statics of a Rigid Body.*

450. Balanced Forces.— A set of external forces which, applied to a rigid body, produces no effect upon its motion, constitutes a *balanced system* or a *system in equilibrium* (Art. 57). The conditions which such a system must satisfy have been considered in Part I under the head of Statics. It may now be shown that the conditions of equilibrium are deducible from the general equations of motion of a rigid body.

451. Equilibrium of Rigid Body.— If all the external forces acting upon a rigid body constitute a balanced system, the body is said to be in equilibrium.

Equilibrium is not synonymous with rest; but a body in equilibrium can have only such motion as it could have if acted upon by no external force.

A single particle acted upon by no force or by balanced forces is either at rest or moving uniformly in a straight line. The possible motion of a rigid body acted upon by balanced forces will be considered below. It will be shown that the individual particles of such a body are not necessarily in equilibrium.

452. Equivalent Systems of Forces.— The equations of motion of a rigid body show that two systems of external forces are equivalent in their effect upon the motion if they satisfy the following conditions:

(1) Their resolved parts* in any direction are equal.

(2) Their moments about an axis containing the mass-center are equal.

For, the second members of equations (1), (2) and (3) of Art. 446 obviously have identical values for two systems satisfying these conditions.

Equivalent forces.—These conditions of equivalence are satisfied by two forces which are equal in magnitude and direction and have the same line of action, even if applied at different points in that line. This is the principle assumed in Art. 82 as one of the fundamental laws of Statics.

---

*By the "resolved part of the system" is meant the sum of the resolved parts of the several forces; by the "moment of the system" is meant the sum of the moments of the several forces.

453. Resultant Force or Resultant Couple.— A system of coplanar forces is in general equivalent to a single force. For, whatever may be the values of X, Y and L for the system, a single force can generally be found having the same values of these three quantities, and thus giving the same equations of motion. Moreover, the values of X, Y and L completely determine the magnitude, direction and line of action of the force.

If the vector sum of the given forces is zero, X and Y are zero whatever the directions of the axes. In such a case there is no single force equivalent to the given system. The simplest equivalent system is then a couple whose moment is equal to the value of L for the system.

454. Conditions of Equilibrium.— In order that a body acted upon by any external forces may move as if acted upon by no force, the values of X, Y and L in the equations of motion must be zero. That is, the conditions of equilibrium for a rigid body are (1) that the sum of the resolved parts of the external forces is zero for each of two rectangular directions, and (2) the sum of the moments is zero for an axis containing the mass-center. It may be shown as in Art. 104 that if these conditions are satisfied,

(1) The sum of the resolved parts in any direction is zero, and

(2) The sum of the moments about any axis is zero.

From these general principles may be derived all the special conditions of equilibrium deduced in Chapter V.

455. Motion Under Balanced Forces.— The nature of the possible motion of a rigid body when the external forces are balanced may be seen from equations (1), (2) and (3), Art. 446. These become

$$d^2x/dt^2 = 0, \qquad \qquad (1)$$

$$d^2y/dt^2 = 0, \qquad \qquad (2)$$

$$d^2\theta/dt^2 = 0. \qquad \qquad (3)$$

Equations (1) and (2) express the fact that the acceleration of the mass-center is zero, and its velocity therefore constant. Equation (3) shows that the angular acceleration is zero and the angular velocity therefore constant.

It is thus seen that if every particle of the body is initially at rest, it will remain at rest. If initially there is a motion of translation, this motion will continue with unvarying velocity. But in general the

motion will be composed of a translation in which the mass-center moves with unvarying velocity, together with a rotation with uniform angular velocity.

The motion of any individual particle is the resultant of the components due to the translation and to the rotation. In general the velocity of a particle not in a line perpendicular to the plane of the motion and containing the center of mass is variable both in magnitude and in direction. Its two components of acceleration may be determined as in Arts. 441, 443. With the notation there used, the translational component p is zero. Of the two components due to the rotation the one perpendicular to r is zero, because the angular acceleration is zero; the other component is $r\omega^2$ directed toward the axis containing the mass-center. This component is therefore the resultant acceleration of the particle. This resultant reduces to zero only for particles lying in an axis containing the center of mass. Except in the special case of translation, therefore, the individual particles not lying in this axis are not in equilibrium.

The acceleration of any particle is determined by the resultant of the forces exerted upon it by other particles of the body. These internal forces, taken as a whole throughout the body, consist of stresses (Art. 36), so that the entire system of internal forces satisfies the same conditions which are satisfied by a system of balanced external forces. But the forces acting upon every individual particle are not balanced, except in the very special case of uniform translatory motion.

CHAPTER XXII.

THE PRINCIPLE OF IMPULSE AND MOMENTUM.

§ 1. *Any System of Particles.*

456. Momentum of a System.— The momentum of a particle has been defined as a vector quantity equal to the product of the mass into the velocity. The *vector sum of the momenta* of all the particles of a system may be called the momentum of the system.

Restricting the discussion to plane motion, and specifying the position of every particle by its coördinates referred to fixed rectangular axes, the axial components of the momentum of a system are

$$m_1\dot{x}_1 + m_2\dot{x}_2 + \quad . \quad . \quad .$$

and

$$m_1\dot{y}_1 + m_2\dot{y}_2 + \quad . \quad . \quad . \quad .$$

But if $\bar{x}, \bar{y}$ are the coördinates of the mass-center and M is the whole mass $m_1 + m_2 + \quad . \quad . \quad .$, we have as in Art. 377,

$$m_1\dot{x}_1 + m_2\dot{x}_2 + \quad . \quad . \quad . \quad = M(d\bar{x}/dt);$$

$$m_1\dot{y}_1 + m_2\dot{y}_2 + \quad . \quad . \quad . \quad = M(d\bar{y}/dt).$$

The total momentum of the system has therefore the same value as if the entire mass were moving with the mass-center.

457. Angular Momentum of a System.— The motion of a particle is at every instant directed along a definite right line in space. Momentum may therefore be regarded as a localized vector quantity of which this line is the *position-line*. The momentum of a particle thus has a definite moment about any origin in the plane of the motion. The *moment of the momentum* of a particle about any point is also called its *angular momentum* about that point (Art. 327).

By the *angular momentum of a system* of particles about any point is meant the sum of the angular momenta of the individual particles about that point.

If the origin of coördinates be taken as origin of moments, the angular momentum of a particle whose mass is m_1 and whose coördinates are x_1, y_1 is

$$m_1(x_1\dot{y}_1 - y_1\dot{x}_1);$$

and the angular momentum of the system is

$$H = m_1(x_1\dot{y}_1 - y_1\dot{x}_1) + m_2(x_2\dot{y}_2 - y_2\dot{x}_2) + \quad . \quad . \quad .$$
$$= \Sigma\, m(x\dot{y} - y\dot{x}).$$

458. External Impulse.— The impulse of a force has been defined in Arts. 315, 316 and 330. Impulse may be described briefly as the *time-integral of a force.* It is a vector quantity, whose direction coincides with that of the force during every elementary interval of time. Any variation in the direction of the force must be taken into account in the integration.

The *impulse of a set of forces* may be defined as the vector sum of their several impulses.

The vector sum of the impulses of all the *external* forces acting upon the particles of a system may be called the *external impulse* for the system.

If X is the sum of the x-components and Y the sum of the y-components of all the external forces, X and Y are also the axial components of the vector sum of the external forces. The axial components of the external impulse during the interval from $t = t'$ to $t = t''$ are therefore

$$\int_{t'}^{t''} X\,dt \quad \text{and} \quad \int_{t'}^{t''} Y\,dt.$$

459. Angular Impulse.— Since a force has at every instant a definite line of action in space, its impulse during an elementary time may be regarded as a localized vector quantity whose position-line is determined by the line of action of the force. Hence the impulse has a definite moment about any origin. The moment of the impulse for any time during which the line of action of the force varies is to be found by adding the moments of the impulses for the elementary intervals.

If L is the moment of a force about any point, its angular impulse for the interval from $t = t'$ to $t = t''$ is

$$\int_{t'}^{t''} L\,dt.$$

For $L\,dt$ is equal to the moment of the impulse of the force for the elementary time dt. (Art. 335.)

The *angular impulse of a set of forces* may be defined as the sum of the angular impulses of the individual forces. If L denotes the sum of the moments of the forces,

$$\int_{t'}^{t''} L\,dt$$

is the sum of the moments of their impulses, or their total angular impulse.

The sum of the angular impulses of all external forces acting upon a system of particles may be called the *external angular impulse* for the system. Its value is

$$\int_{t'}^{t''} L\,dt$$

if L is the sum of the moments of the *external* forces.

460. Time-Integrals of General Equations of Motion.— In Chapter XVIII were deduced three general equations for the plane motion of a system of particles. Two of these (Art. 380) are equations of motion of the mass-center, and one (Art. 383) was called the equation of angular motion. Using the same notation as in Chapter XVIII, except that the total mass is represented by M and the coördinates of the mass-center by x, y, the three equations are

$$X = M(d^2x/dt^2); \quad . \quad . \quad . \quad . \quad (1)$$

$$Y = M(d^2y/dt^2); \quad . \quad . \quad . \quad . \quad (2)$$

$$L = dH/dt. \quad . \quad . \quad . \quad . \quad (3)$$

The first members of these equations depend only upon the *external* forces; X is the sum of their x-components, Y the sum of their y-components, L the sum of their moments about the origin of coördinates. The quantity designated by H is the angular momentum of the system of particles about the origin of coördinates.

Let each equation be integrated with respect to the time between limits $t = t'$ and $t = t''$. Let $\dot{x}', \dot{x}''$ be initial and final values of $\dot{x}$ or dx/dt, with similar notation for $\dot{y}$ and H. Then

$$\int_{t'}^{t''} X\,dt = M(\dot{x}'' - \dot{x}'); \quad . \quad . \quad . \quad (4)$$

$$\int_{t'}^{t''} Y\,dt = M(\dot{y}'' - \dot{y}'); \quad . \quad . \quad . \quad (5)$$

$$\int_{t'}^{t''} L\,dt = H'' - H'. \quad . \quad . \quad . \quad (6)$$

The first member of equation (4) represents the x-component of the total external impulse. The second member is the increment of

the x-component of the total momentum. The members of (5) have similar meanings.

The first member of (6) is the external angular impulse. The second member is the increment of the angular momentum.

Since a vector quantity is completely determined when its axial components are known, equations (4) and (5) express the following proposition:

(1) *The change in the momentum of a system during any interval of time is equal in magnitude and direction to the impulse of the external forces during that interval.*

Equation (6) may be expressed in words as follows:

(2) *The change in the angular momentum of a system during any interval of time is equal to the angular impulse of the external forces for that interval.*

Together these two propositions express the general principle of the *equivalence of impulse and momentum-increment* for any system of particles.

461. Sudden Impulse.— Equations (4), (5) and (6) are wholly general, and may be applied to the solution of any problem in which the expressed integrations can be effected. They are, however, more especially useful when the impulses are sudden (Arts. 319–321). If a very great force acts for a very brief time, the positions of the particles change very slightly during that time. If the changes of position be assumed zero, the application of the equations to particular problems is simplified. In considering the effect of a blow, or the impact of one body against another, the impulse is usually treated as if strictly instantaneous; for most purposes the results are sufficiently near the truth.

In dealing with sudden impulses, the value of the force is usually wholly unknown. The value of the impulse cannot therefore be determined by integration, but can be known only from its effect. For convenience, the integral expressions in equations (4), (5) and (6) may be replaced by single symbols. The equations may be written in the form

$$X' = \Delta(M\dot{x}); \quad \ldots \quad (7)$$

$$Y' = \Delta(M\dot{y}); \quad \ldots \quad (8)$$

$$L' = \Delta H. \quad \ldots \quad (9)$$

Here X' and Y' are the axial components of the total impulse, L' is the total angular impulse:

$$X' = \int_{t'}^{t''} X\,dt; \quad Y' = \int_{t'}^{t''} Y\,dt; \quad L' = \int_{t'}^{t''} L\,dt.$$

The symbol Δ denotes an increment; $\Delta(M\dot{x})$ and $\Delta(M\dot{y})$ are the axial components of the increment of momentum, and ΔH is the increment of the angular momentum.

If the line of action of an instantaneous impulse is known, L' can be expressed in terms of X' and Y'.

462. **Linear Momentum.**— It has been shown that the axial components of the total momentum have the same values as if the entire mass were moving with the mass-center. This is true for any system of particles, whether rigid or not. The effect of an impulse on the motion of the mass-center may therefore be determined as if the system were a single particle. The momentum, computed as if the whole mass were concentrated at the mass-center, may be called the *linear momentum* of the system.

The *angular momentum* has not in general the same value as if the mass were concentrated at the mass-center.

Examples.

1. A body of mass 20 lbs. falls from rest under the action of gravity. Determine, by the principle of impulse and momentum, the velocity of the mass-center after 5 sec.

2. A body whose shape is that of a right circular cylinder, resting with one end upon a smooth horizontal plane, is set in motion by the tension in a flexible cord unwinding from its surface. If the mass of the body is 12 lbs. and the tension is constantly equal to 2 lbs. weight, what will be the velocity of the mass-center after 5 sec.? Does the result depend upon the position of the mass-center in the body?

3. A body of 40 lbs. mass and of any shape, initially at rest, receives a blow equivalent to a force of 100 lbs. acting for ½ sec. What can be determined as to the motion of the mass-center immediately after the blow?

4. Two bodies of masses 10 lbs. and 18 lbs. respectively are connected by a string. Both being initially at rest, the mass of 10 lbs. receives an impulse equivalent to a force of 50 lbs. acting for 1 sec. (*a*) What is the velocity of the mass-center of the system immediately after the impulse? (*b*) If, immediately after the impulse, the mass

of 18 lbs. is at rest, what is the velocity of the mass-center of the other body? (*c*) If, at a certain instant after the impulse, the mass-center of the mass of 18 lbs. has a velocity of 10 ft.-per-sec. in the direction opposite to that of the impulse, what velocity has the mass-center of the other body? *Ans.* (*c*) $18 + 5g = 179$ ft.-per-sec.

5. Two particles whose masses are 5 lbs. and 1 lb. respectively, connected by an inextensible weightless cord 10 ft. long, are initially at the same point. The lighter body is projected vertically upward with a velocity of 40 ft.-per-sec. (*a*) What will be the velocity of their common mass-center immediately after the second particle begins to rise? (*b*) How long will the mass-center continue to rise?

Ans. (*a*) 5.15 ft.-per-sec. (*b*) 0.16 sec.

§ 2. *Rigid Body Having Motion of Translation or of Rotation About a Fixed Axis.*

463. Angular Momentum in Case of Translation.— If the motion of a body is a translation, the motion of every particle is known when that of the mass-center is known. The equation of angular impulse and angular momentum therefore is not needed for the determination of the motion. The value of the angular momentum in case of translation will, however, be found useful.

If v is the translational velocity, the momenta of the several particles are parallel and equal to m_1v, m_2v, ... These momenta form a system analagous to a system of parallel forces applied to the particles and proportional to their masses. The sum of their moments is therefore the same as that of a momentum Mv at the mass-center. That is, in computing the angular momentum of a body whose motion is translatory, the whole mass may be assumed to be concentrated at the center of mass.

464. Linear Momentum of Rotating Body.— Let a rigid body of mass M be rotating, with angular velocity ω, about a fixed axis whose distance from the center of mass is a. Let A (Fig. 183) be the mass-center and O the projection of the fixed axis. The velocity of the mass-center is $a\omega$ perpendicular to OA; hence the linear momentum is $Ma\omega$ perpendicular to OA.

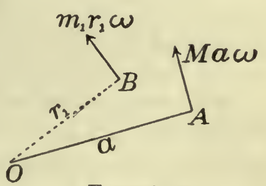


FIG. 183.

465. Angular Momentum of Rotating Body.— The angular momentum about the fixed axis of rotation may be computed as follows:

Let B (Fig. 183) be the position of a particle of mass m_1 distant r_1 from the axis of rotation. Its momentum is $m_1 r_1 \omega$ perpendicular to OB, and the moment of this momentum about O is $m_1 r_1^2 \omega$. The total moment of momentum about O is therefore

$$H = (m_1 r_1^2 + m_2 r_2^2 + \quad . \quad . \quad . \quad) \omega = I\omega,$$

if I is the moment of inertia of the body with respect to the axis of rotation.

466. External Impulse Acting on Rotating Body.— If a body is constrained to rotate about a fixed axis, it is convenient to classify the external forces into *applied forces* and *constraining forces;* the latter being usually wholly unknown, except as the solution of the equations of motion serves to determine them. Similarly, the total external impulse is made up of the *applied impulse* and the *constraining impulse.* The latter is the impulse of the hinge-reaction (Art. 422); it is unknown in magnitude and direction, but it acts through the axis of rotation.

467. Equations of Impulse and Momentum for Rotating Body Acted Upon by Instantaneous Impulse.— Let momentum and impulse be resolved along the line joining the mass-center A to the center of rotation O (Fig. 184). Let P' be the applied impulse, and R' the impulse of the hinge-reaction. Let the components of P' be P'_n in direction AO and P'_t perpendicular to AO; and let the corresponding components of R' be N' and T'. (Fig. 184 represents P'_n and P'_t as if applied at A; this is not necessarily true, but the figure shows correctly the *directions* of the component impulses.)

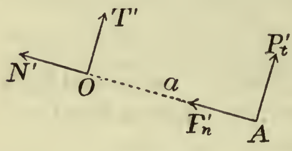


FIG. 184.

Assuming the impulse to be instantaneous, the position of the mass-center A is the same immediately before and immediately after the impulse. The direction of motion of A is therefore not changed by the impulse, but its velocity is changed by $a\Delta\omega$ if $\Delta\omega$ is the increment of the angular velocity. The change of the linear momentum there-

fore has the value $Ma\,\Delta\omega$ directed at right angles to OA; and the linear equations of impulse and momentum take the form

$$P'_t + T' = Ma\,\Delta\omega; \quad . \quad . \quad . \quad (1)$$

$$P'_n + N' = 0. \quad . \quad . \quad . \quad . \quad (2)$$

If L' is the moment of the *external impulse* and I the moment of inertia of the body, both taken about the axis of rotation, the equation of angular impulse and angular momentum takes the form

$$L' = I\Delta\omega. \quad . \quad . \quad . \quad . \quad (3)$$

And since the moment of the *constraining impulse* R' is zero, L' is equal to the moment of the *applied impulse* (Art. 466).

468. Equations of Impulse and Momentum for Rotating Body When Impulse Is Not Sudden.— Equations (1) and (2) are not valid unless the impulse is instantaneous; for otherwise the direction of motion of the mass-center changes during the impulse, and the change in the linear momentum has not the value used in these equations.

Equation (3), however, holds good whatever the time occupied by the impulse; for the increment of the angular momentum is always equal to $I\Delta\omega$, if $\Delta\omega$ is the total increment of the angular velocity. In applying the equation it may be necessary to determine L' by integration.

EXAMPLES.

1. A uniform straight bar, hinged at one end, receives a known sudden impulse at a certain point, in a direction perpendicular to the length. Required to determine (a) the effect on the motion and (b) the constraining impulse.

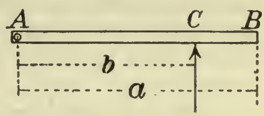


FIG. 185.

Let AB (Fig. 185) be the bar, hinged at A, and let C be the point of application of the impulse.

Let M = mass of bar; a = its length; k = radius of gyration about axis of rotation; ($k^2 = a^2/3$;) P' = magnitude of applied impulse; b = distance AC; ω' = angular velocity just before the impulse; ω'' = angular velocity just after the impulse; $\Delta\omega = \omega'' - \omega'$ = increment of angular velocity.

(a) Equation (3) becomes

$$P'b = Mk^2\Delta\omega,$$

whence $$\Delta\omega = P'b/Mk^2 = 3P'b/Ma^2;$$

from which ω'' can be determined if ω' is known.

(*b*) Since $P'_t = P'$ and $P'_n = 0$, equations (1) and (2) give

$$T' = \tfrac{1}{2}Ma\,\Delta\omega - P' = (3b/2a - 1)P';$$
$$N' = 0.$$

The resultant hinge-impulse is therefore

$$R' = P'(3b - 2a)/2a,$$

its direction being that of P'.

2. Solve Ex. 1 with the following data: Mass of bar = 20 lbs.; arm of impulse = 2 ft.; length of bar = 4 ft.; impulse equivalent to a force of 100 lbs. acting for ½ sec.; initial angular velocity = 0.

Ans. $\Delta\omega = 15g/16$ rad.-per-sec. $T' = -P'/4$.

3. At what point must a uniform bar, hinged at one end, be struck in order that the hinge may receive no shock?

Ans. With notation of Ex. 1, $b = 2a/3$.

4. If a bar of length 3 ft. and mass 24 lbs. is hinged at the middle point and struck at a point 1 ft. from the end with an impulse equivalent to that of a force of 1,000 lbs. acting for 0.1 sec., determine (*a*) the motion just after the impulse and (*b*) the constraining impulse.

Ans. (*a*) $\omega'' = 25g/9$ rad.-per-sec. (*b*) $T' = -P' = -100g$ poundal-sec.

5. A homogeneous circular cylinder is free to rotate about its geometrical axis. A string wrapped about the cylinder is jerked in a direction perpendicular to the axis. If the body receives a given increment of angular velocity, required the value of the impulse applied through the string, and of the impulse of the hinge-reaction.

Ans. $T' = -P' = \frac{1}{2}Ma\,\Delta\omega$, if a = radius of circular section.

6. A homogeneous circular cylinder is free to rotate about its axis of figure, which is horizontal. Around the cylinder is wrapped a string, the free end of which hangs vertically and sustains a body of known mass. This body is lifted vertically and then dropped so that its velocity is V just as the string tightens. Immediately after the impulse its velocity is V'. Required (*a*) the angular velocity of the cylinder just after the impulse, and (*b*) the impulsive tension in the string.

Ans. (*a*) $2m(V - V')/Ma$ rad.-per-sec.; (*b*) $m(V - V')$; M being the mass of the cylinder, m the mass of the other body, a the radius of the cylinder.

469. Center of Percussion.— If a body is free to rotate about a fixed axis, there is a certain line along which an impulse may be applied without causing any hinge-impulse. This may be shown by equations (1), (2) and (3) of Art. 467.

Equation (2) shows that if N' is to be zero, P'_n must be zero; the direction of the applied impulse must therefore be perpendicular to OA (Fig. 184), so that $P'_t = P'$.

Putting $T' = 0$ and $P'_t = P'$ in equation (1),

$$P' = Ma\Delta\omega.$$

If $b =$ arm of P' with respect to the axis of rotation, equation (3) (Art. 467) becomes

$$P'b = I\Delta\omega = Mk^2\Delta\omega.$$

These two equations give

$$b = k^2/a.$$

That is, the line of action of P' must intersect OA produced in a point O' (Fig. 186) such that

$$OO' = k^2/a.$$

This point is called the *center of percussion.*

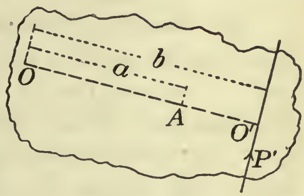


FIG. 186.

Referring to Art. 425, it will be seen that the center of percussion coincides with the center of oscillation of the body if it rotates as a compound pendulum, the fixed axis being horizontal and the applied force being the weight of the body.*

EXAMPLES.

1. A square plate of uniform small thickness and uniform density is free to rotate about one edge as a fixed axis. At what point must it be struck in order that the hinge-impulse shall be zero?

Ans. At a distance from the axis equal to two-thirds the length of a side of the square.

2. The body described in Ex. 1 is free to rotate about an axis passing through one vertex of the square and perpendicular to its plane. Where is the center of percussion?

Ans. At a distance from the axis equal to two-thirds the diagonal.

3. A body is free to rotate about a fixed axis passing through the center of mass. Show that the center of percussion is at an infinite distance from the axis.

* The above discussion assumes that the mass is symmetrically distributed with respect to a plane through the mass-center perpendicular to the fixed axis, and that the impulse is applied in this plane of symmetry. If this condition is not satisfied the impulse will (unless certain very special conditions are satisfied) tend to cause rotation about some axis inclined to the fixed axis, and to resist this tendency impulsive reactions will act at the hinge. See Art. 427.

4. A body free to rotate about a fixed axis passing through the mass-center receives at the same instant two impulses which are equal and opposite but not collinear. Show that the hinge-impulse is zero.

5. A body free to rotate about any fixed axis is acted upon by an impulsive couple (*i. e.*, two impulses as in Ex. 4) whose moment is Q. Determine the hinge-impulse.

Ans. With notation of Art. 467, $N' = 0$, $T' = MaQ/I$.

§ 3. *Resultant Momentum.*

470. General Equation of Impulse and Momentum.— Before taking up the general case of plane motion of a rigid body, it will be well to consider the full meaning of the general principle of the *equivalence of impulse and momentum*, stated in Art. 460.

It was there shown that the total impulse of the external forces during any time and the total change of momentum during that time are equivalent; that is, they are related to each other in the same way as two sets of forces, which, applied to a rigid body, are equivalent in effect. The discussion of Art. 460 was restricted to plane motion, and the principle was proved by deducing three algebraic equations which express the equivalence of two sets of coplanar forces. It will be instructive to give another deduction which is independent of any particular system of coördinates, and has the further advantage of showing the full generality of the principle. Starting with the principle of equivalence of impulse and momentum-increment for a single particle, it may readily be extended to any system of particles.

For a single particle, *the resultant impulse for a time* dt *is equivalent to the change of momentum during that time.* "Resultant impulse" means the impulse of the resultant of all forces, external and internal, which act upon the particle. "Equivalent" means equal in magnitude and direction and having the same position-line. If the impulses for all elementary times throughout a given finite interval be combined as if they were forces, due account being taken of magnitude, direction and position-line, and if the elementary momentum-increments be combined in like manner, the two resultants must be equal.

This principle may be applied to *every individual particle* of a system. Combining the resulting equations for all particles, it fol-

lows that *the resultant impulse for the system is equivalent to the resultant of all the momentum-increments.*

"Resultant impulse" here means the resultant of the impulses of all external and internal forces acting upon the particles; but it may readily be seen that the resultant impulse of the internal forces is zero. This follows from the law of action and reaction. The two forces exerted by any two particles upon each other are at every instant equal, opposite and collinear; therefore their impulses during any time are equal, opposite and collinear. *The impulses of the internal forces for the entire system have therefore a zero resultant.* It follows immediately that

*The resultant external impulse is equal to the resultant change of momentum.**

This general principle is closely analagous to the principle of the equivalence of the external forces and the effective forces (Art. 387). The *equation of impulse and momentum* is in fact the time-integral of the *equation of external and effective forces.* Thus,

$$\text{(resultant external force)} = \frac{d}{dt}\text{(resultant external impulse)};$$

$$\text{(resultant effective force)} = \frac{d}{dt}\text{(resultant momentum)}.$$

471. Algebraic Equations of Impulse and Momentum.—The above general principle is true without restriction to plane motion. To express the principle fully in the case of motion in three dimensions requires six independent equations, just as six equations are required to determine the resultant of a system of forces in three dimensions acting upon a rigid body (Chapter X). When the motion is restricted to a plane, the principle is expressed by three independent equations, just as in the case of coplanar forces in Statics. The three independent equations may be (*a*) two equations of resolution and one of moments, (*b*) one equation of resolution and two of moments, or (*c*) three equations of moments. The restrictions to be observed in choosing origin of moments and direction of resolution are the same as in Art. 104.

In the case of a sudden impulse, the most useful equations are often (7), (8) and (9) of Art. 461. Equations (7) and (8) are ob-

* The remarks made in Art. 387 regarding the meaning of the term "resultant" are applicable here.

tained by resolving along each of a pair of fixed rectangular axes. In certain cases, however, simple equations result by resolving in other directions. In equation (9) the axis of moments is any line perpendicular to the plane of the motion. In order to express the value of H in any case of plane motion, it is necessary to consider the resultant momentum in the general case.

472. Value of Resultant Momentum.— The resultant of a set of coplanar forces is either a single force, a couple, or zero. A similar proposition is true of the *resultant momentum* of any system of particles moving in a plane.

If the vector sum of the momenta of the individual particles is not zero, its value gives the magnitude and direction of the resultant momentum. As in Art. 462, it is seen that in general the resultant momentum has the same magnitude and direction as if the entire mass were moving with the mass-center. Its line of action does not in general contain the mass-center, however, but may be determined by a moment equation.

If the velocity of the mass-center is zero, the vector sum of the momenta of the particles is zero. In this case the resultant momentum is in general a couple, whose moment is the same for every axis, and may therefore be computed for whatever axis is most convenient.

If the velocity of the mass-center is zero and the angular momentum is also zero, the resultant momentum is zero. In this case if the system is rigid it must be at rest; but if not rigid the individual particles are not necessarily at rest.

473. Resultant Momentum in Case of Translation.— In the case of translation, the resultant momentum is a single linear momentum whose value is in all respects (magnitude, direction and position-line) the same as if the entire mass were moving with the mass-center; for the momenta of the particles are proportional to their masses and are parallel.

474. Resultant Momentum of Rigid Body Rotating About a Fixed Axis Containing the Center of Mass.— If a rigid body rotates about an axis containing the mass-center, the resultant momentum is a couple (Art. 472); its moment is equal to the angular momentum about the axis of rotation. The value of this moment is

$I\omega$, if I is the moment of inertia with respect to the axis of rotation and ω the angular velocity (Art. 465).

In this case the angular momentum has the same value for all axes parallel to the axis of rotation.

475. Resultant Momentum of Rigid Body Having Any Plane Motion.— The resultant momentum in the general case of plane motion is best determined by considering the actual instantaneous motion to be made up of two components as in Art. 443:

(*a*) A *rotation*, with the actual angular velocity of the body, about an axis containing the mass-center.

(*b*) A *translation* whose velocity is equal in magnitude and direction to that of the mass-center.

Let I be the moment of inertia with respect to the assumed axis, M the mass, ω the instantaneous angular velocity, and v the instantaneous velocity of the mass-center.

(*a*) The resultant momentum corresponding to the rotation is a couple whose moment is $I\omega$.

(*b*) The resultant momentum corresponding to the translation is a linear momentum Mv whose position-line contains the mass-center.

These can be combined into a single linear momentum equal to Mv at a distance from the mass-center equal to $I\omega/Mv$, which is, therefore, the resultant momentum. In general, however, it is convenient to use the two components (*a*) and (*b*) rather than their resultant.

476. Angular Momentum About Any Axis.— The angular momentum about any axis perpendicular to the plane of the motion may be found by taking the sum of the moments of the two components of the resultant momentum. The moment of the couple of course has the same value $I\omega$ for every axis.

The angular momentum about any axis perpendicular to the plane of the motion is equal to the angular momentum about a parallel axis containing the mass-center plus the angular momentum of the entire mass assumed concentrated at the mass-center.

§ 4. *Any Plane Motion of a Rigid Body.*

477. Equations of Impulse and Momentum in General Case of Plane Motion.— We are now prepared to apply the equations of

Art. 461 to the unrestricted plane motion of a rigid body. It must be remembered, however, that the equations may take other forms, depending upon the choice of the coördinates of position and of the axis of moments. The application of the equations may best be explained by the solution of specific problems. For convenience, the general equations are here repeated.

$$X' = \Delta(M\dot{x}); \quad . \quad . \quad . \quad . \quad (1)$$

$$Y' = \Delta(M\dot{y}); \quad . \quad . \quad . \quad . \quad (2)$$

$$L' = \Delta H. \quad . \quad . \quad . \quad . \quad (3)$$

In equation (3), moments are taken about *any* axis perpendicular to the plane of motion. If the axis contains the mass-center, $H = I\omega$, and the equation becomes

$$L' = I\Delta\omega. \quad . \quad . \quad . \quad . \quad (4)$$

This is identical in *form* with equation (3) of Art. 467, which applies to the case of rotation about any fixed axis; but in that case I is the moment of inertia about the axis of rotation.

478. Effect of Instantaneous Impulse on Free Body Initially at Rest.— Let a body, initially at rest but free to move in a plane, receive an instantaneous impulse of known magnitude, direction and line of action.

Since the resultant momentum is zero before the impulse,

(momentum-increment) = (resultant momentum after impulse).

Hence the motion just after the impulse must be such that the resultant momentum has the same magnitude, direction and position-line as the impulse. Let P' be the magnitude of the impulse, b the distance of its line of action from the mass-center, $M =$ mass of body, $Mk^2 = I =$ its moment of inertia with respect to the axis through the mass center, $v =$ velocity of mass-center, and $\omega =$ angular velocity just after the impulse. Then by Art. 475 the resultant momentum is equal in magnitude to Mv, its direction is that of v, and its moment about the mass-center is $Mk^2\omega$. Hence the direction of v coincides with that of P', and

$$P' = Mv, \quad P'b = Mk^2\omega;$$

or

$$v = P'/M, \quad \omega = P'b/Mk^2.$$

Instantaneous axis.— Let A (Fig. 187) be the mass-center, and draw AB perpendicular to the line of action of P'. Since the velocity

of A after the impulse is parallel to P', the instantaneous motion is a rotation about some axis C intersecting BA produced. Let $AC = z$; then $v = z\omega$, or

$$z = v/\omega = k^2/b.$$

The points C and B are thus related in the same way as the centers of suspension and oscillation of a compound pendulum (Art. 425).

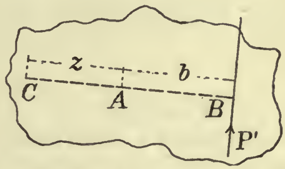


FIG. 187.

The determination of the instantaneous axis and of the angular velocity gives a clear idea of the state of motion immediately after the impulse. If the body is acted upon by no forces subsequently, the angular velocity remains constant, and the velocity of the mass-center remains constant in magnitude and direction; the instantaneous axis therefore continues at the same distance from the mass-center, and its locus is a plane parallel to the path of the center of mass. The surface traced in the body by the instantaneous axis is a circular cylinder of radius z whose geometrical axis contains the mass-center.

Application of equations.— To show the application of the equations of Art. 477, let the axis of x have the direction of the impulse. Then

$$X' = P', \quad Y' = 0;$$

and if moments are taken about the axis through the mass-center,

$$L' = P'b.$$

Equations (1), (2) and (4) therefore become

$$M\Delta\dot{x} = P', \quad M\Delta\dot{y} = 0, \quad Mk^2\Delta\omega = P'b.$$

But $\Delta\dot{x}$ and $\Delta\dot{y}$ are the components of v, and since $\Delta\dot{y} = 0$,

$$v = \Delta\dot{x} = P'/M.$$

Also, $\Delta\omega$ becomes in this case ω, hence

$$\omega = P'b/Mk^2,$$

as before.

To illustrate still further the general method, let moments be taken about a point in the line of action P'. The initial angular momentum about this point is zero, the moment of P' is zero, hence

the angular momentum about this point after the impulse is zero. But by Art. 476 its value is

$$Mk^2\omega - Mvb;$$

equating this to zero,

$$v/\omega = k^2/b,$$

which agrees with the result already found. This moment equation may take the place of one of the equations before used.

EXAMPLES.

1. A straight bar of uniform small cross-section and uniform density, free to take up any plane motion, receives a blow at a given point directed at right angles to its length. Determine the instantaneous axis of rotation.

Ans. With the above notation, $z = l^2/12b$, l being the length of the bar.

2. In Ex. 1, let the mass of the bar be 20 lbs., its length 2 ft., and let the impulse be equivalent to that of a force of 1,000 lbs.-weight acting for 0.1 sec. If $b = l/2$, determine v and ω after the impulse. *Ans.* $v = 5g$ ft.-per-sec; $\omega = 15g$ rad.-per-sec.

3. A straight bar of uniform cross-section and uniform density, 2 ft. long, receives a blow in a direction perpendicular to its length at a point 6 ins. from one end. Determine the instantaneous axis.

Ans. 4 ins. from end.

4. In Ex. 1, let the direction of the impulse be inclined at angle θ to the length of the bar. Determine the instantaneous axis.

5. If, in Ex. 3, the impulse is applied at the same point but in a direction inclined 30° to the bar, determine the instantaneous axis.

Ans. 4 ins. from middle point of bar.

6. A square plate of uniform small thickness and uniform density is suspended by strings attached to two corners so that two edges are vertical. It receives a blow perpendicular to its plane, applied at the middle point of the lowest edge. Determine the instantaneous axis. *Ans.* If a = side of square, $z = a/6$.

7. A circular plate of uniform density and uniform small thickness is suspended by a string attached at a point in the circumference. How must it be struck in order that it shall begin to rotate about a vertical tangent? *Ans.* At a point bisecting a horizontal radius.

8. If the plate suspended as in Ex. 7 receives a blow in a direction perpendicular to its plane and at a distance from the center equal to one-third the radius, determine the instantaneous axis.

Ans. $z = 3a/2$, if a = radius.

9. A homogeneous parallelopiped of mass 10 kilogr. and dimensions (in c.m.) $20 \times 30 \times 40$, receives a blow whose direction is

perpendicular to one of the largest faces and whose point of application is the middle point of its shorter edge. Determine the instantaneous axis. If the velocity of the mass-center after the impulse is 1 met.-per-sec., determine the angular velocity and the value of the impulse.

Ans. $z = 8\frac{1}{3}$ c.m.; $\omega = 12$ rad.-per-sec.; $P' = 10^6$ dyne-seconds.

10. In the above general problem, discuss the case $b = 0$.

11. Discuss the case in which b is infinite while $P'b$ is finite.

12. A horizontal blow must be applied at what point of a billiard ball in order that it shall begin to roll without sliding?

479. Effect of Impulsive Couple on Free Body Initially at Rest.— If a body is subjected to two instantaneous impulses which are equal in magnitude but opposite in direction and not collinear, the motion of the body immediately after the impulse must be such that the resultant momentum is a couple. This requires that the instantaneous axis shall pass through the mass-center.

If L' is the moment of the impulsive couple, the angular velocity after the impulse is

$$\omega = L'/Mk^2.$$

EXAMPLES.

1. A uniform straight bar of length l receives simultaneously two blows applied close to the ends, at right angles to the length, and in opposite directions. Either blow alone, applied at the middle point, would give the bar a translational velocity V. Determine the motion after the impulse.

Ans. A rotation about a central axis, with angular velocity $12V/l$.

2. A uniform straight bar of length l receives simultaneously two impulses P' and $2P'$ at right angles to the length and in the same direction, the former applied at the middle, the latter at one end. The impulse P' alone would give the bar a translational velocity V. Determine the instantaneous motion.

Ans. A rotation with angular velocity $12V/l$ about an axis distant $l/4$ from the middle point.

3. A uniform bar 2 ft. long whose mass is 10 lbs., constrained to rotate about one end, receives an impulse at a point 0.25 ft. from the free end which gives it an instantaneous angular velocity of 360° per sec. An equal impulse is applied to an exactly similar bar which is wholly free. Determine the instantaneous motion.

Ans. An angular velocity of $24\pi/7$ rad.-per-sec. about an axis $5\frac{1}{3}$ ins. from the middle point.

4. A body is free to rotate about a fixed axis through the mass-center. Show that an impulsive couple will cause no hinge-impulse.

480. Effect of Instantaneous Impulse on Free Body Not Initially at Rest.—The *momentum-increment* due to a given impulse has the same value, whatever the initial motion of the body. The motion immediately after the impulse is found by combining the motion due to the impulse with the motion previously existing.

Thus if $\dot{x}, \dot{y}$ are the axial components of the velocity of the mass-center and ω the angular velocity, and if values just before the impulse are denoted by a single accent and values just after the impulse by a double accent,

$$\dot{x}'' = \dot{x}' + \Delta\dot{x},$$

$$\dot{y}'' = \dot{y}' + \Delta\dot{y},$$

$$\omega'' = \omega' + \Delta\omega;$$

in which $\Delta\dot{x}$, $\Delta\dot{y}$ and $\Delta\omega$ are to be determined from equations (1), (2) and (4) of Art. 477.

The velocity of the mass-center after the impulse is equal to the *vector sum* of its velocity before the impulse and that due to the impulse. The angular velocity after the impulse is the *algebraic sum* of the angular velocity before the impulse and that due to the impulse.

Change of instantaneous axis.—The position of the instantaneous axis of rotation immediately after the impulse is less simply determined in case the body is in motion before the impulse than when it is initially at rest. It may, however, be determined, by combining the change of motion due to the impulse with the initial motion.

Let the change of motion due to the impulse, computed as if the body were initially at rest, be a rotation about an instantaneous center C (Fig. 188) with angular velocity ω; let the motion just before the impulse be a rotation about an instantaneous center C' with angular velocity ω'; and let the motion immediately after the impulse be a rotation about an instantaneous center C'' with angular velocity ω''. By Art. 437 C'' lies upon the line CC' at such a point that

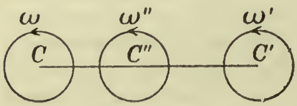


Fig. 188.

$$CC'' \times \omega = C'C'' \times \omega',$$

and
$$\omega'' = \omega + \omega'.$$

If ω' and ω have like signs, C'' lies between C' and C; if they have opposite signs, C'' lies on $C'C$ produced.

If ω' and ω are equal and opposite, C'' is at infinity on CC'. In this case ω'' is zero, and the resultant motion is a translation.

EXAMPLES.

1. A free circular disc of uniform density and thickness is rotating about a central axis perpendicular to its plane when it receives a blow directed along a tangent. What is the angular velocity after the impulse if the velocity of the mass-center is v''?

Let r = radius of disc, ω' its original angular velocity, P' the impulse. From the given data,

$$P' = Mv'',$$

and the moment of P' about the mass-center is

$$L' = Mv''r$$

in magnitude, but may be either positive or negative. If positive, the equation of moments about the mass-center is

$$Mv''/r = \tfrac{1}{2}Mr^2\Delta\omega,$$

whence $\quad \Delta\omega = 2v''/r; \quad \omega'' = \omega' + \Delta\omega = \omega' + 2v''/r.$

If L' is negative, $\quad \omega'' = \omega' - 2v''/r.$

2. A rotating wheel falls vertically in its own plane and strikes a horizontal plane which is perfectly rough and inelastic. Determine the subsequent motion. ["Perfectly rough" means that no sliding occurs; "perfectly inelastic" means that the wheel does not rebound but remains in contact with the plane.]

This problem may be solved by a single equation obtained by taking moments about an axis through the point of contact of the wheel with the plane. The moment of the impulse about this axis is zero, hence the angular momentum has the same value before and after the impulse. Letting v'' denote the velocity of the mass-center after the impulse, and taking other notation as usual, the angular momentum before the impulse is

$$Mk^2\omega'.$$

After the impulse it is

$$Mk^2\omega'' + Mv''r,$$

if r is the radius of the wheel. Therefore

$$Mk^2\omega' = Mk^2\omega'' + Mv''r.$$

But also, $v'' = r\omega''$; hence

$$\omega'' = \omega'k^2/(k^2 + r^2); \quad v'' = r\omega'k^2/(k^2 + r^2).$$

3. A circular disc of uniform thickness and density, of radius 2 ft. and mass 50 lbs., rotating at the rate of 100 revolutions per minute, falls in its own vertical plane and strikes a horizontal plane surface. At the instant of striking its center has a horizontal velocity of 20 ft.-per-sec. and a vertical velocity of 30 ft.-per-sec. If the plane is so rough that there is no sliding and so inelastic that there

is no rebound, determine (*a*) the subsequent motion, (*b*) the value of the impulse.

Ans. (*a*) $v'' = 20.32$ ft.-per-sec. or -6.36 ft.-per-sec., depending upon the direction of v'. (*b*) The horizontal impulse is either 16 poundal-seconds or $-1,318$ poundal-seconds; the vertical impulse is 1,500 poundal-seconds.

4. A homogeneous right circular cylinder, resting with one end upon a smooth horizontal plane, is set in motion by a jerk applied to the free end of a string wrapped on its surface. Determine the instantaneous axis of rotation.

Ans. At a distance from the geometrical axis equal to half the radius.

5. In Ex. 4, let the radius of the cylinder be 0.5 met. and its mass 5 kilogr. If the angular velocity just after the impulse is 2 rev.-per-sec., what is the value of the impulse? What is the velocity of the mass-center?

Ans. Impulse $= 250,000\pi$ dyne-sec. Velocity of mass-center $= 50\pi$ c.m.-per-sec.

6. A body has any plane motion, when a certain line in the body, perpendicular to the plane of the motion, suddenly becomes fixed. Determine the subsequent motion.

The impulse by whose action the stated change in the motion is produced acts in some line passing through the axis which becomes fixed. The angular momentum with respect to this axis is therefore unchanged. The motion after the impulse may be determined by equating the values of the angular momentum before and after the impulse.

7. A homogeneous right circular cylinder is rotating freely about its axis of figure when an element of the cylindrical surface suddenly becomes fixed. Determine the subsequent motion.

Let M be the mass of the cylinder, a its radius, ω' its original angular velocity, ω'' its angular velocity after the impulse. The original value of the angular momentum about an axis instantaneously coinciding with an element of the surface is $Ma^2\omega'/2$. The angular momentum about this axis after the impulse is $3Ma^2\omega''/2$. Hence $\omega'' = \frac{1}{3}\omega'$.

8. In Ex. 7, determine the value of the impulse.

The linear velocity of the mass-center changes from 0 to $a\omega''$; hence the value of the impulse is $Ma\omega''$, its direction being perpendicular to the plane containing the fixed axis and the geometrical axis.

9. A uniform straight bar of mass 10 lbs. and length 2 ft., rotating about its mass-center at the rate of 5 rev.-per-sec., receives a blow equivalent to a force of 100 lbs. acting for 0.5 sec. at a point 3 ins. from one end, in a direction perpendicular to the length of

the bar. Determine the instantaneous center and angular velocity just after the blow.

Ans. Instantaneous center is 7.10 ins. from middle point of bar.

10. With the initial motion described in Ex. 9, (*a*) what impulse will reduce the motion to a translation of 10 ft.-per-sec.? (*b*) What impulse will leave the body at rest?

Ans. (*a*) An impulse of 100 poundal-seconds, whose line of action is $\frac{1}{3}\pi$ ft. from the mass-center.

11. Show that an impulsive couple, applied to a body rotating about an axis containing the mass-center, does not change the axis of rotation.

12. A body whose mass is 10 kilogr. and least radius of gyration 15 c.m. has at a certain instant a translational velocity of 2 met.-per-sec. and an angular velocity of 2 rev.-per-sec. Two opposite impulses are simultaneously applied in lines 10 c.m. apart, each equivalent to a force of 5,000 dynes acting for 10 sec. Determine the subsequent motion.

Ans. The linear velocity is unchanged; the angular velocity is increased or decreased by 2/9 rad.-per-sec.

481. Connected Bodies.—The principle that the external impulse is equal to the change of momentum it produces may be applied to any number of bodies regarded as forming a system. If the bodies press against one another, or are connected by strings or by hinges, the forces they exert upon one another are internal, and the impulses of these forces may be omitted from the equations of impulse and momentum for the system. For plane motion three independent equations can therefore be written which do not involve the impulses of these internal forces or "reactions." These are not generally sufficient to determine the motion. To complete the solution it is necessary to write equations for single bodies of the system. For any given body it is often possible to form one or more such equations not containing the unknown reactions.

The method of procedure will be illustrated by examples.

EXAMPLES.

1. Two uniform straight bars AB, BC, connected by a frictionless hinge at B, rest on a horizontal plane so that ABC is a straight line. The bar AB receives at a certain point a horizontal blow at right angles to its length. Determine the motion just after the impulse.

Let P' denote the impulse, applied at distance a from B (Fig. 189). Let m, m' be the masses and l, l' the lengths of AB and BC respectively. Since the velocity of the mass-center of the system

just after the impulse will be parallel to P', it is obvious that every point of both bars will instantaneously move parallel to P'. Let v be the velocity of the mass-center of AB and ω its angular velocity, v', ω' being the velocity of the mass-center of BC and its angular velocity. Then v' can be expressed in terms of v, ω, ω', as follows:

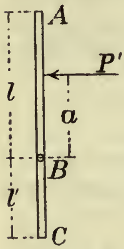


FIG. 189.

If u is the velocity of B,

$$u = v - \tfrac{1}{2}l\omega;$$

$$v' = u - \tfrac{1}{2}l'\omega' = v - \tfrac{1}{2}(l\omega + l'\omega').$$

Equating the impulse P' to the linear momentum of the system after the impulse,

$$P' = mv + m'v'$$

$$= (m + m')v - \tfrac{1}{2}m'(l\omega + l'\omega'). \qquad (1)$$

Taking moments about B for the body AB, thus eliminating the hinge-impulse,

$$P'a = ml^2\omega/12 + mvl/2. \quad . \quad . \quad . \quad (2)$$

Taking moments about B for the body BC,

$$0 = m'l'^2\omega'/12 - m'v'l'/2 = (3l\omega + 4l'\omega' - 6v)m'l'/12. \quad (3)$$

Equations (1), (2) and (3) serve to determine v, ω and ω' when P' is known. Without knowing P', the *ratios* of ω, ω' and v can be determined.

2. If the two bars are equal in all respects and the impulse is applied at the middle point of AB, show that $\omega = \omega' = 6v/7l$.

3. At what point must the impulse be applied in order that the bar BC shall be instantaneously at rest? Will it remain at rest?

4. In Ex. 1, suppose the two bars equal in all respects and the impulse applied at A. Show that $\omega' = -\omega/3 = -6v/5l$.

5. Two bodies A, B of masses m, m', are connected by a string which coincides with the line joining their mass-centers. At a certain instant they are moving with equal velocities at right angles to the string, when A receives an impulse in the direction BA which, if A were free, would have deflected it 45°. Determine the motion just after the impulse, assuming that the string does not stretch nor break.

6. The initial conditions being as in Ex. 5, let A receive an impulse directed at an angle of 30° from BA produced, of such magnitude that if A were free is would be deflected 15°. Determine the motion just after the impulse.

Ans. If v is the original velocity, the bodies have equal velocities $[0.3169m/(m + m')]v$ in the direction BA after the impulse; the velocity of B perpendicular to BA is unchanged, while the final velocity of A perpendicular to BA is $1.183v$.

CHAPTER XXIII.

THEORY OF ENERGY.

§ 1. *External and Internal Work.*

482. **Work of a Stress.**— The forces which any two particles exert upon each other are equal and opposite, constituting a stress (Art. 36). The total work done by the two forces of a stress during any displacements of the particles upon which they act may be computed as follows:

Let A and B (Fig. 190) be the two particles, r their distance apart at any instant, P the magnitude of the force which each exerts upon the other. Assume that the force acting upon A has the direction BA; then the force acting upon B has the opposite direction AB.

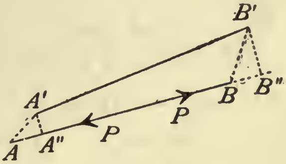


FIG. 190.

Let the particles receive any infinitesimal displacements AA', BB', such that the angle turned through by AB is also infinitesimal, and let AA'', BB'' be the orthographic projections of these displacements upon AB. Then

$$-P \times AA'' = \text{work done by force } P \text{ acting on } A;$$

$$P \times BB'' = \text{“ “ “ “ } P \text{ “ “ } B;$$

$$P(BB'' - AA'') = \text{total work done by stress.}$$

But $\quad BB'' - AA'' = A''B'' - AB = A'B' \cos (d\theta) - AB,$

if $d\theta$ is the angle between $A'B'$ and AB. And since

$$\cos (d\theta) = 1 - \tfrac{1}{2}(d\theta)^2 + \ . \ . \ . \ ,$$

$A'B' \cos (d\theta)$ differs from $A'B'$ by an infinitesimal of the second order, and

$$A'B' \cos (d\theta) - AB = A'B' - AB = dr.$$

The total work done by the stress during the supposed infinitesimal displacement is therefore Pdr; and during any finite displacement the work is equal to

$$\int_{r'}^{r''} P\,dr, \quad . \quad . \quad . \quad . \quad (1)$$

r' and r'' being the initial and final values of r.

The same result may be reached by another method, as follows: Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the rectangular coördinates of the particles A and B respectively, and let α, β, γ be the angles between AB and the axes. By Art. 352, the work done by either of the forces P is equal to the sum of the works done by its axial components. Hence the force P acting upon A does an amount of work

$$-\int P \cos \alpha \cdot dx_1 - \int P \cos \beta \cdot dy_1 - \int P \cos \gamma \cdot dz_1,$$

and the force P acting upon B does an amount of work

$$\int P \cos \alpha \cdot dx_2 + \int P \cos \beta \cdot dy_2 + \int P \cos \gamma \cdot dz_2;$$

the limits of the integrations being so taken as to correspond to the initial and final positions of the particles. The total work done by both the forces of the stress is therefore

$$\int P[(dx_2 - dx_1) \cos \alpha + (dy_2 - dy_1) \cos \beta + (dz_2 - dz_1) \cos \gamma].$$

From the equation

$$r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$

there results, by differentiation,

$$r\,dr = (x_2 - x_1)(dx_2 - dx_1) + (y_2 - y_1)(dy_2 - dy_1) + (z_2 - z_1)(dz_2 - dz_1);$$

from which, since

$$(x_2 - x_1)/r = \cos \alpha, \quad (y_2 - y_1)/r = \cos \beta, \quad (z_2 - z_1)/r = \cos \gamma,$$

we have

$$dr = (dx_2 - dx_1) \cos \alpha + (dy_2 - dy_1) \cos \beta + (dz_2 - dz_1) \cos \gamma.$$

The above value of the total work done by the two forces of the stress therefore reduces to

$$\int_{r'}^{r''} P\,dr,$$

as before.

Two particles rigidly connected.—If the two particles A and B are rigidly connected, or if they move in such a way that the distance between them remains constant, the work done by the stress is zero; for if r is constant, the definite integral (1) is zero, whatever the value of P, so long as it is finite.

The above reasoning obviously holds for any two equal and opposite forces applied to two particles and directed along the line joining them, whether the two forces constitute a stress, or whether they are exerted by other bodies or particles.

483. External and Internal Work Defined.— Work done upon a system of particles is *external* if done by an external force (a force exerted by a particle not belonging to the system); it is *internal* if done by an internal force.

484. Configuration.— A definite set of relative positions of the particles of a system is called a *configuration*. If all the distances between the particles remain constant, the configuration remains constant. If the configuration does not change, the system either remains at rest or moves as if the particles were rigidly connected.

485. Internal Work Done Only During Change of Configuration.— Every internal force is one of the forces of an internal stress. The work done by a stress is zero unless the particles between which the stress acts change their distance apart (Art. 482). The total work done by the internal forces upon the particles of a system is therefore zero unless the configuration changes.

§ 2. *Energy of Any System of Particles.*

486. Energy of a System Defined.*— When the condition of a system is such that it can do work against external forces, it is said to possess energy.

The quantity of energy of a system is the quantity of external work it will do in passing from its present condition to some standard condition.

The meaning of "condition" will be made clear by the following discussion of the method of estimating the quantity of energy.

487. Kinetic Energy of a System of Particles.— It has been shown (Art. 358) that a particle of mass m and velocity v possesses energy of motion or kinetic energy to the amount $\frac{1}{2}mv^2$, the "standard condition" being one of rest. A system of particles of masses

* The meaning of energy of a system has already been explained in an elementary manner in Chapter XVII (Arts. 360–366). Some of the definitions and principles there given are here repeated in order to make the following more rigorous discussion complete.

m_1, m_2, ... , having velocities v_1, v_2, ... , possesses an amount of kinetic energy

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \quad . \quad . \quad . \quad .$$

For if the particles are brought to rest by the action of external forces, this amount of external work will be done.

488. **Energy of Configuration, or Potential Energy.**— If any particle of a system is acted upon by both external and internal forces, it may do *external* work without any change of velocity. For its velocity will remain unchanged so long as the internal work done upon it is equal to the external work done by it.

It thus appears that a system may be able to do external work (and therefore may possess energy) by reason of the internal forces. Since the total internal work is zero for any displacement which leaves the configuration unchanged, this form of energy depends upon the possibility of changing the configuration. It may therefore be called *energy of configuration.* The name *potential energy* is, however, more generally used.

489. **Meaning of Standard Condition.**— The meaning of the word "condition" used in the definition of energy now becomes clear. By a definite condition of a system is meant a definite configuration together with a definite set of velocities.

In order to estimate the total amount of energy possessed by a system in a given condition, it is necessary to assume a standard configuration and a standard set of velocities; and to compute the total quantity of external work done by the particles while the system passes from the given condition to this standard condition.

In estimating the kinetic energy of a particle, any velocity may be taken as the standard, but it will here be assumed zero for every particle. This simplifies the algebraic expression for kinetic energy, and does not detract from the generality of the discussion.

The choice of the standard configuration will be governed by convenience in each particular case.

490. **Total Energy of a System.**—The foregoing principles will now be expressed in definite mathematical form.

Let it be required to compute the total external work done by a system in passing from any given condition to an assumed standard condition.

Let v_1 = velocity of particle of mass m_1,

I_1 = work done upon the particle by internal forces,

E_1 = work done by the particle against external forces.

If the particle is brought to rest,

$$E_1 - I_1 = \tfrac{1}{2}m_1v_1^2.$$

A similar equation may be written for every particle; that is,

$$E_2 - I_2 = \tfrac{1}{2}m_2v_2^2;$$

$$\cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot$$

By addition,

$$(E_1 + E_2 + \ \ldots\ ) - (I_1 + I_2 + \ \ldots\ )$$
$$= \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2 + \ \ldots\ .$$

Or, if

$$E = E_1 + E_2 + \ \ldots\ ,$$
$$I = I_1 + I_2 + \ \ldots\ ,$$
$$K = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \ \ldots\ ,$$

the equation may be written

$$E = I + K.$$

Since E denotes the total external work done by all the particles while the system passes to the standard condition, it is by definition (Art. 486) the *total energy of the system* in its initial condition. Of the two parts I and K whose sum is the total energy, the second is the *kinetic energy* of the system, being equal to the sum of the kinetic energies of the individual particles. The part I is the total work done by the internal forces while the system passes to the standard condition. This is the *energy of configuration* or *potential energy*. In regard to this quantity an important question must be raised.

491. Is Energy a Definite Quantity?—A system may, in general, change from its present condition to the standard condition in different ways. Can it be assumed that the quantity of external work done by the system will be the same, whichever one of the possible ways is actually followed? If different ways of making the change result in different quantities of external work, the energy of the system, as the term has been defined, cannot have a definite value.

The kinetic energy obviously has a definite value, depending only upon the velocities of the particles and their masses. It remains to consider the potential (or configuration) energy. And since the po-

tential energy is equal to the total work done by the internal forces during the change to the standard configuration (Art. 490), the question reduces to this: *Does the total work done by the internal forces depend only upon the initial and final configurations, or does it depend upon the way in which the change of configuration takes place?*

In the case of any actual material system, this question can be answered only by experience. The foregoing discussion has referred to ideal systems consisting of material particles exerting upon one another forces which, for any two particles, act along the line joining the particles, and follow Newton's law of action and reaction. No assumption has been made as to whether the magnitude of the stress acting between any two particles depends upon their distance apart, or upon their relative velocities, or upon other conditions.

It is possible, without contradicting any dynamical law heretofore assumed, to assign laws of force which shall make the internal work depend only upon the total change of configuration; it is equally possible to assign laws which shall make the internal work depend upon the way in which the change of configuration takes place.

492. Conservative and Non-conservative Systems.— Ideally, then, two kinds of material systems may exist.

If the total internal work done during any change of configuration depends only upon the initial and final configurations, the system is called *conservative*.

If the total internal work depends upon the way in which the change of configuration is made, the system is *non-conservative*.

It is only for conservative systems, as thus defined, that energy is a definite quantity.

493. Law of Force in Conservative System.— It may be shown that if the stress acting between any two particles of the system is directed along the line joining the particles, and if its magnitude depends only upon their distance apart, the system is conservative.

It was shown in Art. 482 that the work done by the stress between any two particles is equal to

$$\int_{r'}^{r''} P\,dr,$$

if P is the magnitude of each of the equal and opposite forces of the stress, r the distance between the particles, r' and r'' the initial

and final values of r. If P is a function of r, $\int P\,dr$ is a function of r, and the definite integral is a function of r' and r''. Since the internal forces consist wholly of stresses, the total internal work is a function of the initial and final values of the distances between the particles; that is, it depends only upon the initial and final configurations. The assumed law of force therefore makes the system conservative.

The above assumption is not, however, the only one which makes the system conservative. Considering all possible pairs of particles, let P_1 be the magnitude of the stress acting between a pair whose distance apart is r_1; P_2 the magnitude of the stress between a pair whose distance apart is r_2; etc.; and let I denote the total work done by the internal forces during any given change of configuration. Then (Art. 482)

$$dI = P_1 dr_1 + P_2 dr_2 + \quad . \quad . \quad . \quad .$$

If $P_1, P_2, \ . \ . \ .$ are such functions of $r_1, r_2 \ . \ . \ .$ that the second member of this equation is the differential of a function of $r_1, r_2, \ . \ . \ . \ ,$ the integration of the equation between proper limits gives I as a function of the initial and final values of $r_1, r_2, \ . \ . \ . \ ,$ that is, as a function of the initial and final configurations.

As an illustration, a system of three particles is conservative if the law of force is expressed by the equations

$$P_1 = A + B(r_2 + r_3) + Cr_2r_3;$$
$$P_2 = A + B(r_3 + r_1) + Cr_3r_1;$$
$$P_3 = A + B(r_1 + r_2) + Cr_1r_2;$$

A, B and C being constants.

Examples.

1. Two particles attract each other with a stress of constant magnitude equal to 100 poundals. What is the potential energy of the system when the particles are 10 ft. apart, if in the standard configuration they are 4 ft. apart?

2. Two particles attract each other with forces varying inversely as the square of their distance apart, the value of the attraction being c when the distance is a. If the distance is b in the standard configuration, what is the potential energy of the system when the distance is r? *Ans.* $ca^2(1/b - 1/r)$.

3. A system consists of three particles attracting one another according to the law of the inverse square of the distance. The attraction between any two when 1 met. apart is 5,000 dynes. In

the standard configuration their distances are 2, 3 and 4 met. respectively. Determine the value of the potential energy when each of the three distances is 10 met. *Ans.* 3.92×10^5 ergs.

494. Principle of Work and Energy for Conservative System.— Let a conservative system experience any change of condition, not necessarily passing to the standard condition. The initial and final conditions may be designated briefly as C_1 and C_2, and the standard condition as C_0. In passing from C_1 to C_0 *in any way* the total external work is a definite quantity E_1, equal by definition to the energy of the system in the condition C_1. In passing from C_2 to C_0 *in any way* the total external work is a definite quantity E_2, equal to the energy of the system in the condition C_2. Hence in passing from C_1 to C_2, the external work done by the system is a definite quantity $E_1 - E_2$. That is,

The total external work done by a conservative system during any change of condition is equal to the decrease of its total energy.

The same principle may be stated in the following form :

The total work done upon a conservative system by external forces during any change of condition is equal to the increase of its total energy.

As a particular case, suppose a change in which no external work is done. The total energy must remain constant, the potential and kinetic energies receiving opposite and equal changes.

In the following Section will be given a brief discussion of the theory of energy as applied to the actual systems of nature.

§ 3. *Conservation of Energy.*

495. Transfer of Energy from One System to Another.— When a conservative system does work against external forces, it loses an amount of energy equal to the work done. In certain cases this loss is accompanied by a gain of energy on the part of some other body or system.

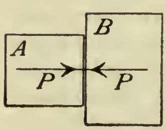


FIG. 191.

Thus, consider two bodies A and B (Fig. 191) to be in contact and to be exerting upon each other equal and opposite forces at the surface of contact, the magnitude of each force being P, and its direction being normal to the surface of contact. Let the bodies receive displacements whose components parallel to

the forces are equal, each being equal to h, and their direction being such that A does positive work and B therefore does negative work. The body A loses energy to the amount Ph, and the body B gains an equal amount. The action between the two bodies has therefore resulted in no change in their total amount of energy; one has gained exactly as much as the other has lost.*

Again, consider two rough bodies A and B in contact (Fig. 192), exerting upon each other forces which have components both normal and tangential to the surface of contact. The normal components are equal and opposite forces N, and the tangential components are equal and opposite forces T. Let both bodies be displaced parallel to the forces T; let h' be the displacement of A and h'' the displacement of B, their directions being the same, and h' being greater than h''. No work is done by or against either of the normal forces. The forces T act in such directions as to resist the sliding of one body over the other; that is, the direction of the force T acting upon A is opposite to the direction of the displacement of its point of application, while the force T acting upon B acts in the direction of the displacement of its point of application.

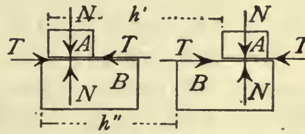


FIG. 192.

The body A does work equal to $+ Th'$, and loses energy equal to $+ Th'$.

The body B does work equal to $- Th''$, and gains energy equal to $+ Th''$.

The body A therefore loses more energy than B gains. The quantity of energy $T(h' - h'')$ has apparently disappeared.

If, in the last case, $h' = h''$, so that no sliding occurs, the energy gained by B is equal to that lost by A. If $h'' = 0$, B gains no energy, and there is an apparent loss of all the energy given up by A.

Another example of an apparent loss of energy is furnished by a moving body suddenly brought to rest, as a falling body striking the earth. The kinetic energy possessed by the body just before striking is wholly lost, without any apparent gain by other bodies.

---

* It should be noticed that nothing is here said of the relation of A or of B to other bodies. The energy received by B from A may be continually transferred, in whole or in part, to other bodies.

An example of the transfer of energy from one system to another is furnished by a body moving upward against the force of gravity. A body of mass m, moving upward with velocity v, possesses kinetic energy equal to $\frac{1}{2}mv^2$. As it rises, its kinetic energy decreases and finally is wholly lost. Is this loss accompanied by a gain on the part of other bodies or systems? The other body concerned is the earth; obviously the system consisting of the earth and the body has gained an amount of potential energy equal to mgh, if h is the vertical distance ascended by the body. If, during this ascent, the velocity of the body changes from v_1 to v_2, its loss of kinetic energy is $\frac{1}{2}m(v_1^2 - v_2^2)$. But mgh and $\frac{1}{2}m(v_1^2 - v_2^2)$ are equal (Art. 227); that is, the gain of potential energy by the system of the earth and the body is exactly equal to the loss of kinetic energy by the body.

The last example is an illustration of what occurs whenever a conservative system suffers a change of configuration without doing external work. The individual members of the system may gain or lose kinetic energy, but the system at the same time loses or gains an equal amount of potential energy. If, in the supposed case, the earth and the body be regarded as a system, their energy does not vary in total amount, but merely changes, in part from kinetic to potential.

To summarize: When a body or a system loses energy, there may be (a) an equal gain by other bodies or systems; (b) a gain by other systems apparently less than the loss by the given system; or (c) no apparent gain by other systems.

In some cases, then, there seems to be a disappearance of energy.

496. Equivalence of Energy and Heat.— In many cases in which energy disappears, heat is generated, and it is found that there is a definite relation between the quantity of energy that disappears and the quantity of heat that is generated. Taking as the unit quantity of heat the amount which will raise the temperature of a pound of water one degree Fahr. (the British thermal unit), the disappearance of 778 foot-pounds of energy is accompanied by the generation of 1 unit of heat.*

---

*To make the unit heat definite, the temperature of the water should be specified, since the quantity of heat required to raise the temperature of a pound of water one degree is not exactly the same at all temperatures. The numerical value of the energy-equivalent of the unit heat, expressed in foot-

In other cases, as in the steam-engine, energy comes into existence while heat disappears; one unit of heat being consumed for every 778 foot-pounds of energy generated.

From these facts it is natural to conclude that *heat and energy are equivalent*, and that a given quantity of one may be *transformed* into an equivalent quantity of the other.

497. **Forms of Energy.** — Modern science goes further and regards heat not merely as equivalent to energy but as being actually a form of energy. Moreover, heat is but one of several forms of energy into which the ordinary kinetic and potential energy possessed by bodies and systems may be converted, either directly or indirectly. The meaning of the word energy is thus enlarged; and we recognize mechanical energy, heat energy, chemical energy, electrical energy.

Mechanical energy. — Energy which depends upon the configuration of a material system or upon the motions of its constituent particles is mechanical energy. The discussion included in Arts. 486–494 refers solely to this kind of energy.

Heat energy. — The reasons for regarding heat as a form of energy are given above.

Chemical energy. — The chemical combination of two bodies is often accompanied by the generation of heat. In this case there is found to be a definite relation between the amount of chemical change and the quantity of heat generated. Thus, if one pound of water is produced by the combination of hydrogen and oxygen, the quantity of heat generated is 6,900 British thermal units. Again, by the expenditure of this amount of heat (or of its equivalent in other forms of energy), a pound of water may be decomposed into hydrogen and oxygen. In such a case as this the two bodies when uncombined may be said to possess chemical energy.

Electrical energy. — Another important form of energy is possessed by bodies by reason of their electrical condition. If a body

---

pounds, varies with the value of the pound-force, which varies with the locality. The value 778 foot-pounds is sufficiently correct for most purposes. Expressed in British absolute units of work (foot-poundals) the energy-equivalent of the unit heat is approximately 25,000.

Using French units, the quantity of heat required to raise the temperature of a kilogram of water a degree Cent. is equivalent to about 427 meter-kilograms of energy.

is charged with electricity, and if the conditions are such that there can be a flow of electricity accompanied by a fall of potential,* the body possesses electrical energy. If the flow actually takes place, the electrical energy is transformed into energy of some other form. In the case of an electric current, there is a continuous flow of electricity and a fall of potential in the direction of the flow. A part of the electrical energy is transformed into heat which manifests itself in the heating of the conductor carrying the current. A part may be transformed into mechanical energy by means of a motor. Or the current may be employed in decomposing a chemically compound substance as water into its elements, a part of the electrical energy being then transformed into chemical energy.

Nature of different forms of energy.— The nature of each of the above forms of energy is unknown; any hypothesis concerning it must involve a theory as to the ultimate constitution of matter. A plausible supposition is that heat and chemical energy are dependent upon the motions and relative positions of the ultimate particles of which bodies are made up. On this supposition, these forms of energy do not differ in nature from the kinetic and potential energy possessed by any system by virtue of its configuration and of the visible motions of its parts. Thus, the accepted theory regarding the nature of heat is that it is *kinetic energy possessed by the molecules* of bodies; and a possible explanation of the development of heat during the chemical combination of two substances is that there is a transformation of mechanical energy from potential into kinetic. The atoms of hydrogen and those of oxygen, when brought into certain relations, may exert upon each other forces, under the action of which the particles approach one another or assume definite relative positions; during this process work may be done upon the particles by these forces, and their kinetic energy may thus be increased, the molecular kinetic energy being heat. Before the combination the two bodies, regarded as a system, possess energy by virtue of the relative positions of their particles and of the forces exerted between them. If this is the true explanation, the energy, both before and after the combination takes place, is of the same

---

* No explanation of the meaning of electrical potential can be given here. The above brief statement regarding electrical energy presupposes an elementary knowledge of the theory of electricity.

nature as ordinary mechanical energy; being at first potential, and then transformed into kinetic.

Attempts have been made to explain electrical energy as mechanical energy possessed by the hypothetical "ether."

The supposition that all energy is really mechanical may or may not become more probable with the advance of knowledge; the fact that the different forms of energy are equivalent in the sense that a definite quantity of one can be converted into a definite quantity of another is independent of any hypothesis as to their nature.

498. **Conservation of Energy.**— One of the most important results of modern science is the establishment of the principle that *no system of bodies gains or loses energy wihout an equivalent loss or gain by other bodies or systems.*

499. **Conservation of Energy in Machines.**— A machine is a structure designed to do work against external forces.

In doing external work, a machine loses an amount of energy equal to the work done. Energy must therefore be supplied to it if it is to continue to do work indefinitely. The operation of a machine thus involves (a) the transfer of energy to the machine from other bodies, and (b) the transfer of energy from the machine to other bodies. In general, these two processes go on simultaneously at nearly the same rate. If they go on at unequal rates, the quantity of energy possessed by the machine fluctuates.

If energy is received by the machine and given off by it at the same rate, the machine is a body or system in equilibrium. In this case the relations which must hold among the forces acting upon it may be determined by the principles of Statics. Machines have been considered from this point of view in Part I. The theory of energy furnishes a means of dealing with machines in a more general manner.

In the light of the principle of the conservation of energy, a machine may be defined as *a structure designed to transfer energy from one body or system to another.*

500. **Utilized Energy and Lost Energy.**— That part of the energy given up by a machine which is equivalent to the useful work done may be called *utilized* energy. The energy given up by the machine can never be wholly utilized. In order that the parts of the machine may move in the desired manner, they must be guided by other bodies; these exert frictional forces, against which work must

be done. In doing work against the frictional forces, energy is transformed into heat. Such energy may be called *lost*, since no useful effect can be obtained from it.

The external forces acting upon a machine may be classified (as in Art. 111) as *efforts*, *resistances* (*useful* and *prejudicial*), and *constraints*. In terms of work and energy these may be defined as follows:

An *effort* is a force which does positive work upon the machine. The machine receives a quantity of energy equal to the work done.

A *resistance* is a force which does negative work upon the machine (or against which the machine does positive work). The machine gives up a quantity of energy equal to the work done. Energy given up in doing work against useful resistances is *utilized*; it is *lost* if given up in doing work against prejudicial resistances.

A *constraint* is a force which does no work, positive or negative, upon the machine. The office of a constraint is to guide the motion of some machine part. No energy is gained or lost by the machine by reason of a constraint.

501. Equation of Energy for a Machine.— The transformations of energy with which a machine is concerned during a given time may be expressed algebraically, in the the most general case, as follows:

Let $W =$ work done upon the machine by the efforts
$=$ energy received by the machine;
$W' =$ work done by the machine against useful resistances
$=$ energy utilized;
$W'' =$ work done by the machine against prejudicial resistances
$=$ energy lost;
$E =$ increase in the quantity of energy possessed by the machine.

Then
$$W = W' + W'' + E.$$

If the energy possessed by the machine remains constant, the equation is
$$W = W' + W''.$$

The same equation holds if the interval is so chosen that the energy possessed by the machine has the same value at the end as at the beginning of the interval, even though it has fluctuated during the interval.

502. Efficiency.— For the efficient working of a machine, the energy lost is to be made as small as possible.

The *efficiency* of a machine is the ratio of the energy utilized to the energy received by the machine, the energy stored in the machine being supposed to remain constant (or the interval of time to be so taken that the *total* energy lost by the machine has equaled the total energy received).

Consider an interval such that the energy stored in the machine has the same value at the beginning and at the end of the interval, so that $E = 0$. The efficiency may be expressed by the equation

$$e = W'/W = W'/(W' + W'').$$

If W'' could be made zero, the efficiency would be 1; its actual value is always less than 1.

EXAMPLES.

1. The fuel used in running a steam-engine is coal of such composition that the combustion of 1 lb. produces heat sufficient to raise the temperature of 12,000 lbs. of water 1° Fahr. It is found that $3\frac{1}{2}$ lbs. of fuel are consumed per horse-power per hour. What is the efficiency of the entire apparatus? *Ans.* 0.061 nearly.

2. A steam-engine uses coal of such composition that the combustion of 1 lb. generates 10,000 British thermal units. If 40 lbs. of coal are used per hour, and if the efficiency is 0.08, what horse-power is realized?

3. A dynamo is driven by an engine working as described in Ex. 2. If its efficiency is 0.78, what "activity" in kilowatts is represented by the current generated?

§ 4. *Principle of Work and Energy Applied to Rigid System.*

503. Potential Energy of Rigid System.— It was shown in Art. 485 that internal work is done only when the configuration of the system changes. In any possible displacement of a *rigid* system, therefore, the internal work is zero.

It follows that *the potential energy of a rigid system is zero.*

504. Kinetic Energy of Rigid System.— When the condition of motion of a rigid system at any instant is known, the value of the kinetic energy can be expressed in a simple manner.

(*a*) *Translation.*— In case of translation the velocities of all particles are equal in magnitude. The kinetic energy is therefore

$$K = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \quad . \quad . \quad . \quad = \tfrac{1}{2}Mv^2, \quad . \quad (1)$$

if M is the total mass of the system and v the velocity of every particle.

(*b*) *Rotation about fixed axis.*—Let $r_1, r_2, \ . \ . \ .$ be the distances from the axis of rotation of particles whose masses are $m_1, m_2, \ . \ . \ .$; I the moment of inertia of the system with respect to the axis of rotation; ω the angular velocity at any instant. The velocities of the several particles have values $r_1\omega, r_2\omega, \ . \ . \ .$, and the kinetic energy is therefore

$$K = \tfrac{1}{2}m_1(r_1\omega)^2 + \tfrac{1}{2}m_2(r_2\omega)^2 + \quad . \quad . \quad .$$
$$= \tfrac{1}{2}(m_1r_1^2 + m_2r_2^2 + \quad . \quad . \quad . \quad)\omega^2;$$

or $$K = \tfrac{1}{2}I\omega^2. \quad . \quad . \quad . \quad . \quad (2)$$

(*c*) *Any plane motion.*—Since any plane motion of a rigid body is at every instant equivalent to a rotation about a definite axis, the kinetic energy in any case (except translation) is given by equation (2), if I is the moment of inertia with respect to the instantaneous axis. If the axis of rotation is fixed, I is constant; but in general I is variable.

Let M = total mass of system;
a = distance of mass-center from instantaneous axis;
I_0 = moment of inertia with respect to central axis parallel to instantaneous axis;
$v = a\omega$ = velocity of mass-center.

From Art. 400, $$I = I_0 + Ma^2;$$

therefore $$K = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}I_0\omega^2 + \tfrac{1}{2}Mv^2. \quad . \quad . \quad . \quad (3)$$

The first term of this value of K would be the value of the kinetic energy of the system if rotating, with its actual angular velocity, about a fixed axis through the mass-center. The second term would be the value of the kinetic energy if every particle had a velocity equal to that of the mass-center. These two quantities may be called *rotational* energy and *translational* energy respectively.

505. Principle of Work and Energy for Rigid System.—Since the potential energy of a rigid system is zero, the principle of work and energy takes the following form:

The total work done $\left\{ \begin{matrix} \textit{by} \\ \textit{upon} \end{matrix} \right\}$ *a rigid system is equal to the* $\left\{ \begin{matrix} \textit{decrease} \\ \textit{increase} \end{matrix} \right\}$ *of its kinetic energy.*

506. Equation of Work and Energy in Case of Translation.— Let W denote the total work done upon a rigid system by all external forces during a certain interval, and let v' and v'' be the initial and final values of the velocity of the mass-center. If the motion is a translation, the equation of work and energy is

$$W = \tfrac{1}{2}M(v''^2 - v'^2), \quad . \quad . \quad . \quad (4)$$

which is identical with the equation of work and energy for a particle of mass M moving with the center of mass of the system.

The same equation applies to *any case in which the angular velocity* remains constant; for equation (3) shows that if ω is constant the whole change in K is equal to the change in the term $\frac{1}{2}Mv^2$. Equation (4) may therefore be applied to *any case in which the resultant of the external forces is a single force whose line of action passes through the mass-center of the body;* since the angular velocity is constant when this condition is fulfilled (Art. 444).

In the case of translation, the work done by any force may be computed as if the force were applied at the mass-center, since its actual point of application receives, in any interval, a displacement equal and parallel to that of the mass-center.

In the case of uniform angular velocity, the *actual displacement* of the point of application of each force must be used * in computing W.

507. Equation of Work and Energy in Case of Rotation.— If a body rotates about a fixed axis, equation (2) gives the value of the kinetic energy at any instant. If ω' and ω'' are the initial and final values of the angular velocity, the equation of work and energy may be written in the form

$$W = \tfrac{1}{2}I(\omega''^2 - \omega'^2). \quad . \quad . \quad . \quad (5)$$

508. Computation of Work in Case of Rotation.— The work done by any force upon a rotating body may be expressed in terms of the moment of the force and the angular displacement of the body.

Let A (Fig. 193) be the point of application of a force, O being the center of the circle described by A. Let P = magnitude of force; r = length OA; L = moment of P about the axis of rotation; ϕ = angle between OA and P; θ = angle between OA and some fixed line in the plane of motion.

---

* See, however, Art. 509.

If the body receives an infinitesimal angular displacement $d\theta$, the displacement of A is $r\,d\theta$, and its direction is perpendicular to OA. The work done by the force is therefore

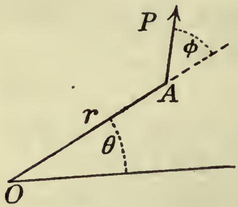


FIG. 193.

$$dW = Pr \sin\phi \cdot d\theta.$$

But $$L = Pr \sin\phi;$$

hence $$dW = L\,d\theta.$$

That is, for an infinitesimal displacement, the work done is equal to the product of the moment of the force into the angular displacement.

For a finite displacement, let θ' and θ'' be the initial and final values of θ; then

$$W = \int_{\theta'}^{\theta''} L\,d\theta.$$

If L remains constant during the displacement,

$$W = L(\theta'' - \theta').$$

509. Equation of Work and Energy in Any Plane Motion.— Let a rigid system, restricted to plane motion, be acted upon by any forces, and receive any displacement during a certain interval. Let M, I, p, v, ω, ϕ, θ have the same meanings as in Arts. 443, 446; let P denote the vector sum of the applied forces, and L the sum of their moments about the mass-center.

The equation of motion of the mass-center is (Art. 445)

$$P = Mp,$$

being identical with the equation of motion of a particle of mass M acted upon by a force P. Resolving P and p along the tangent to the path described by the mass-center, and integrating the resulting equation exactly as in Art. 353, there results the equation

$$W_1 = \tfrac{1}{2}M(v''^2 - v'^2), \qquad . \quad . \quad . \quad (6)$$

in which v' is the initial and v'' the final value of v, and W_1 is the total work done by the applied forces *on the assumption that they act at the mass-center.*

The equation of angular motion is (Art. 446)

$$L = I(d\omega/dt).$$

Multiplying this equation member by member by the equation $d\theta = \omega\, dt$,

$$L\, d\theta = I\omega\, d\omega.$$

Integrating, $$\int_{\theta'}^{\theta''} L\, d\theta = \tfrac{1}{2}I(\omega''^2 - \omega'^2),$$

in which ω' and ω'' are the initial and final values of ω, and θ', θ'' the initial and final values of θ.

The first member of this equation is equal to the work which would be done by the actual forces if the rotation occurred about a fixed axis through the mass-center (Art. 508). Calling this quantity W_2,

$$W_2 = \tfrac{1}{2}I(\omega''^2 - \omega'^2). \quad . \quad . \quad . \quad (7)$$

Referring to the value of the total kinetic energy of a rigid system, given in equation (3), it is seen that the values of W_1 and W_2 given by equations (6) and (7) are equal respectively to the increments of the two portions of the kinetic energy which have been called *energy of translation* and *energy of rotation*. Hence the following important principle may be stated:

In any plane motion of a rigid system, the increment of the energy of translation is equal to the work done by the external forces computed as if they were applied at the mass-center; the increment of the rotational energy is equal to the work done by the external forces computed as if the rotation occurred about a fixed axis through the mass-center.

EXAMPLES.

1. A homogeneous cylinder of mass 20 lbs. and radius 6 ins. rotates about its axis of figure at the rate of 300 rev.-per-min. Required the value of the kinetic energy in foot-pounds.

2. The mass of a wheel-and-axle is 50 lbs., its radius of gyration with respect to the axis of rotation is 10 ins., and the radius of the axle is 6 ins. It is set rotating by a constant tension of 5 pounds-force in the rope which unwinds from the axle. Required the angular velocity of the body after making 4 rev., starting from rest.

Ans. 1.72 rev.-per-sec.

3. Take data as in Ex. 2, except that the tension is due to a weight of 5 lbs. suspended from the rope. (*a*) Determine the angular velocity of the body after making 4 rev. (*b*) Determine the tension in the rope. [Neglect the weight of the rope in both cases.]

Ans. (*a*) 1.69 rev.-per-sec. (*b*) 4.91 lbs.

4. A body of any shape is projected upward against gravity; determine the height to which the mass-center will rise, if no external force acts except gravity. Solve by the principle of work and energy.

[Notice that the reasoning by which equation (6) was deduced is valid without restricting the motion to a plane.]

5. Apply the equation of work and energy to a compound pendulum, and deduce equation (5) of Art. 425.

6. Solve Ex. 9, Art. 447, by the principle of work and energy.

7. Solve Ex. 10, Art. 447, by the principle of work and energy.

§ 5. *The Principle of Virtual Work.*

510. Equation of Virtual Work for a System of Particles.— It has been shown (Art. 369) that if a particle in equilibrium be assumed to receive any arbitrary small displacement, the total work done upon it by all forces is equal to zero. By simple addition this principle may be extended to any system of particles. That is,

If every particle of a system is in equilibrium, the total work done upon all the particles during any arbitrary small displacements is zero.

In general the "equation of virtual work" obtained by applying this principle involves both internal and external forces. In many important cases, however, the displacements may be so taken as to eliminate certain of the internal forces.

511. Equation of Virtual Work for Rigid Body.— Since the total internal work is zero for a displacement which leaves the configuration unchanged, the internal forces may be omitted from the equation of virtual work for a rigid body. That is,

If a rigid body is in equilibrium, the total work done by the external forces during any arbitrary displacement is zero.

In applying this principle, it is generally necessary to take the displacement infinitely small, since a finite displacement will so change the relation among the lines of action of the forces that they will no longer be in equilibrium.

Since the negative of any one of a number of forces in equilibrium is the resultant of all the rest, and since reversing a force reverses the sign of the work done by it without otherwise changing the value, it follows that *the total quantity of work done by any number of external forces acting upon a rigid body during any displacement is equal to the work done by their resultant.*

512. Equations of Equilibrium for a Rigid Body.— The general equations of equilibrium for a rigid body, already deduced in Part I, may be obtained by applying the principle of virtual work.

Thus, if a body be conceived to receive an arbitrary small *translation* in a given direction, the virtual work of any force is equal to the product of the displacement into the resolved part of the force in the direction of the displacement; and the total virtual work is equal to the product of the displacement into the sum of the resolved parts of the forces in the direction of the displacement. Since this total work is zero, whatever the direction of the displacement, it follows that, for equilibrium, *the algebraic sum of the resolved parts of the forces in any direction is zero.*

Again, let the virtual displacement be a rotation about a fixed axis M. The work done by any force is equal to the product of the angular displacement into the moment of the force about M (Art. 508); and the total virtual work is equal to the angular displacement into the sum of the moments of the forces about M. Since this total work is zero, whatever the position of the axis of rotation, it follows that, for equilibrium, *the sum of the moments of the forces about any axis is equal to zero.*

These two general forms include all the equations of equilibrium for forces applied to a rigid body, whether in two dimensions or in three, as already given in Chapter X.

513. Connected Bodies.— If the principle of virtual work is applied to a system of rigid bodies connected in any manner, as by hinges or strings, or pressing against one another, the equation of work must in general include (*a*) all external forces acting upon any of the bodies and (*b*) the forces exerted by the bodies upon one another by reason of their connections. In certain important cases, however, forces due to connections may be omitted from the equation because it can be seen that their work vanishes.

Thus, any two adjacent portions of a tense string exert upon each other forces constituting a tensile stress. If the distance between the two portions remains constant, the total work of the stress is zero (Art. 482). But if the length of the string changes, the total work of the internal forces will not be zero. If a system includes two bodies which are connected by a string, the tension in this string will not enter the equation of virtual work for the system if the dis-

placement is such that the string remains tight and of unchanged length.

The same is true of the contact-stress between two bodies touching each other, if the surfaces of contact are smooth and the displacement is such as not to destroy the contact. Thus, the action and reaction between two bodies connected by a smooth hinge joint may be omitted from the equation of virtual work. But if the surfaces are rough and the displacement is such that sliding occurs, the frictional work must enter the equation.

514. Solution of Problems in Equilibrium.— The principle of virtual work furnishes a simple solution of many problems relating to the equilibrium of rigid bodies, or of connected systems of bodies. The method of applying the principle is to assume an arbitrary infinitesimal displacement, compute the work done by all the applied forces, and equate the total work to zero. By choosing different displacements, different equations may be obtained. In general, for a rigid body acted upon by coplanar forces, three independent equations may be formed.

Any force may be eliminated by taking the displacement of its point of application perpendicular to the direction of the force. By this means two external forces acting on a rigid body may be eliminated, since the directions of displacement of two points may be chosen arbitrarily.

515. Simple Machines Treated by Principle of Work.— The principle of work often furnishes the most convenient method of determining the relation between effort and load in the case of a simple machine or a combination of simple machines.

System of pulleys.— To determine the relation between effort and load in case of a system of pulleys arranged as in Fig. 194.

The system consists of two sets of pulleys, each set comprising three pulleys mounted freely upon a common axis and free to rotate independently of one another; and a cord passing continuously around the pulleys as shown, one end being attached at A, and the effort P being applied at the other end. The axis of the set A is firmly attached to a fixed support C, while the load W is applied to B. This "load" may be the weight of a heavy body, or it may be a force applied horizontally or in any other direction.

Assuming the cord to be perfectly flexible and inextensible, and

neglecting axle friction, the principle of work may be applied as follows:

Let the system be displaced in such a way as to shorten the distance AB by an amount h, the cord remaining tense. The straight portions of the cord being practically parallel to AB, the end of the cord moves a distance $6h$. The effort P and the resistance W are the only forces entering the equation of work for the system composed of the two sets of pulleys and the cord, and we have

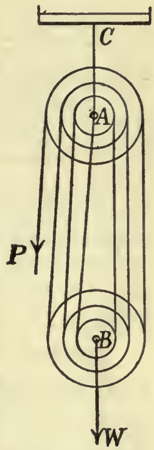


FIG. 194.

$$\text{work done by effort} = 6Ph;$$
$$\text{" \quad " \quad " load} = -Wh.$$

The equation of work is therefore

$$6Ph - Wh = 0; \quad \therefore \quad P = W/6.$$

Train of toothed wheels.— The principle of work may be applied without difficulty to any connected series of toothed wheels, one of which is acted upon by a force tending to produce rotation, and another by a counterbalancing force.

Fig. 195 represents three pairs of wheels, freely mounted upon fixed axes at A, B and C, so connected by teeth that an angular displacement of one must be accompanied by a proportional angular displacement of each of the others. A force P, applied as shown, tends to produce a displacement in one direction, while Q tends to produce an opposite displacement.

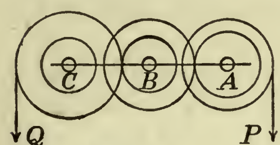


FIG. 195.

The relation between the displacements of any two points of the system may readily be determined from the dimensions and manner of connection. Neglecting axle friction and friction at the surfaces of contact of the teeth, the equation of work for the system shown will contain no forces except P and Q. If a and b are the distances moved by the points of application of P and Q respectively, the equation of work is

$$Pa - Qb = 0.$$

Pulleys and belts.— Let motion be transmitted from one set of machinery to another through a pair of pulleys, rigidly connected

and mounted freely upon a common axis, each connected by a belt with another pulley from which it receives (or to which it imparts) motion by means of the belt. (Fig. 196.)

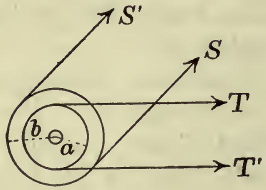


FIG. 196.

Assuming the belts to be perfectly flexible and inextensible, and neglecting axle friction, the equation of work may be written as follows:

Let a and b be the radii of the two pulleys, T and T' the tensions in the two portions of one belt, S and S' the tensions in the two portions of the other. Then, for any angular displacement $\Delta\theta$, the equation of work is

$$Ta\Delta\theta - T'a\Delta\theta - Sb\Delta\theta + S'b\Delta\theta = 0;$$

or
$$(T - T')a = (S - S')b.$$

Screw.— A screw, used as a machine, is arranged to work in a fixed nut, and is usually provided with an arm rigidly attached to the screw and at right angles to its length. The *effort* is applied at the end of this arm and perpendicularly to it and to the axis of the screw, while the *load* is applied at the end of the screw, parallel to its length. The load may be the weight of a heavy body which is to be lifted, the fixed nut being supported upon a solid foundation in the earth.

Neglecting friction, the relation between load and effort may be found as follows:

Let $P =$ effort, applied at distance a from the axis of rotation of the screw, in a direction perpendicular to the arm and to the axis of the screw; $W =$ load, applied in a direction parallel to the axis of rotation; $b =$ pitch of screw (distance between centers of two consecutive threads, measured parallel to the axis of rotation).

Let the screw and arm rotate at a uniform rate, so that the external forces are in equilibrium, and let the work done by all forces acting upon the body be computed for one revolution.

The work done by P is $2\pi aP$; that done by W is $-Wb$. The forces exerted by the fixed body do no work, if friction is neglected.

The total work done upon the body is

$$2\pi aP - Wb = 0;$$

whence
$$P = Wb/2\pi a.$$

By making b small, W may be made very great in comparison with P.

The following examples are designed to illustrate the application of the principle of virtual work. They are all problems in Statics, and may also be solved by the methods of Part I.

EXAMPLES.

1. Determine the position of equilibrium of a bar resting with both ends against the inner surface of a smooth hemispherical bowl.

If the bar be displaced from the position of equilibrium in such a way that the ends slide along the surface of the bowl, the work done by the reactions is zero; hence the work done by gravity (the only other external force) must be zero. This requires that the displacement of the center of gravity shall be horizontal. In order that this requirement shall be fulfilled for every possible sliding displacement, the center of gravity must be vertically below the center of the bowl.

2. Find the position of equilibrium of a heavy bar resting with its ends upon two smooth inclined planes.

Let AB (Fig. 197) represent the bar, C being its center of gravity, and let it receive a displacement such that the ends slide upon the planes. Reasoning as in Ex. 1, it is seen that, if the bar is displaced from the position of equilibrium, the displacement of C must be horizontal if the ends slide along the planes, since the work done by the weight of the bar must then be zero. From this condition and the geometry of the figure, the position of equilibrium may be determined.

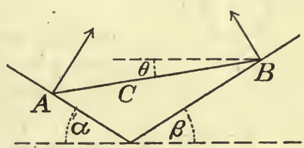


FIG. 197.

Let θ be the inclination of the bar to the horizontal, and y the vertical distance of the center of gravity above a horizontal plane through the line of intersection of the supporting planes. Let α and β be the inclinations of the planes to the horizontal, $AC = a$, $CB = b$. Let y be expressed as a function of θ; the result is

$$y = (a + b)\frac{\sin \alpha}{\sin (\alpha - \beta)} \sin (\beta - \theta) + a \sin \theta.$$

As the bar passes through the position of equilibrium while sliding on the planes, $$dy/d\theta = 0.$$

Differentiating and reducing,

$$(a + b) \tan \theta = a \operatorname{cotan} \alpha - b \operatorname{cotan} \beta.$$

3. A bar 7 ins. long, whose mass-center is 3 ins. from one end, rests within a smooth hemispherical bowl 12 ins. in diameter. What is the inclination of the bar to the vertical when in equilibrium?

Ans. 84° 9′.

4. A bar 12 ins. long, whose mass-center is 7 ins. from one end, rests upon two planes inclined 30° to the horizontal. Determine the position of equilibrium. *Ans.* $\theta = 16° 6'$.

5. A straight bar rests with one end upon a smooth inclined plane and leans against a smooth peg. Determine the position of equilibrium.

6. A bar leans against a smooth peg, the lower end resting upon a smooth surface. Determine the form of the surface in order that the bar may be in equilibrium wherever placed.

Ans. A section of the surface by a vertical plane is represented by the polar equation $r = a + c \sec \theta$.

7. A uniform bar AB, of weight G, can turn freely in a vertical plane about a hinge at A. To B is attached a string which passes over a smooth peg at C and sustains a weight P. Determine the position of equilibrium.

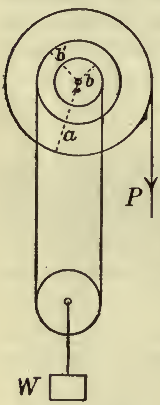


FIG. 198.

8. A screw of ¼ in. pitch is used to raise a weight of 5,000 lbs. Neglecting friction, what force must be applied at the end of an arm 3 ft. long?

9. Determine, by the principle of work (neglecting friction), the relation between the effort (P) and the load (W) in case of the differential wheel-and-axle (Fig. 198).

Ans. $P/W = (b' - b)/2a$.

10. In Ex. 9, let the radius of the wheel be 2 ft. and the radii of the two portions of the axle 6 ins. and 4 ins. If the load is 1,000 lbs., what must be the value of the effort?

11. A string AB is attached to a uniform bar BC of which the end C rests against a smooth vertical plane AC. Find the position of equilibrium.

Ans. If the inclinations of the string and bar to the vertical are θ and ϕ respectively, $\tan \phi = 2 \tan \theta$ in the position of equilibrium.

12. Two uniform bars, AB, BC, of unequal lengths and masses, are connected by a smooth hinge at B. At A and C are small rings which slide on a smooth horizontal wire. Determine the position of equilibrium.

Ans. There will be equilibrium if either AB or BC is vertical. One of these solutions is imaginary.

13. Take data as in Ex. 12, except that the bars are not uniform, so that their mass-centers are not equally distant from the wire. Determine the position of equilibrium.

14. Two equal bars connected by a smooth hinge rest upon a smooth cylinder whose axis is horizontal. Determine the position of equilibrium.

Ans. Let $a =$ distance of mass-center of each bar from hinge, $r =$ radius of cylinder, $\theta =$ angle between either bar and the vertical in the position of equilibrium. Then $a \sin^3\theta - r \cos\theta = 0$.

15. Two particles of unequal masses, connected by a weightless inextensible string, rest on the surface of a smooth cylinder whose axis is horizontal. Determine the position of equilibrium.

CHAPTER XXIV.

RELATIVE MOTION.

§ 1. *Meaning of Absolute and Relative Motion.*

516. Necessity of a "Base" for Specifying Motion.—Attention has been called (Art. 267) to the fact that, in order to specify the motion of a particle, a reference body must be chosen, and that the values of the velocity and the acceleration of a particle estimated with respect to one body are in general different from their values estimated with respect to another. The body to which, in any given case, the motion is referred, may conveniently be called the *base.*

For the purposes of mathematical analysis, the base is replaced by certain lines forming a *geometrical frame.* Thus, for specifying motions with respect to the earth, a frame may be chosen consisting of three rectangular axes intersecting at some definite point on the earth and having directions fixed with reference to the earth.

It is, however, obvious that a frame need not be fixed to any single body in order to serve as a base for specifying motions. For example, the directions of the lines forming the frame may be specified with respect to the fixed stars.

517. "True" and "Apparent" Motion.—When the motions of bodies near the earth are observed, it is with reference to the earth that those motions are commonly specified. But in certain cases the rotation of the earth about its axis is brought into consideration, and we say that the motion of a body referred to the earth as base is only its "apparent" motion; and that the motion of the earth itself must be taken into account if it is desired to determine the "true" motion of the body.

It is, however, obvious that in taking account of the earth's rotation what we are really doing is to refer the motions of terrestrial bodies to a new base. And it is necessary to raise the question whether there is any reason for regarding the motion referred to this new base as the true motion, or even as being a nearer approximation to the true motion than is obtained by taking the earth as

base. Is it, in fact, possible to assign any meaning to the "real" motion of a body as distinguished from its "apparent" motion?

It is evident that, so long as our object is to study the motions of bodies *kinematically*, any base whatever may serve the purpose. What base shall be chosen in any particular case is purely a matter of convenience. From this point of view there is no reason for drawing a distinction between true and apparent motions. Or, as certain writers express it, one base gives as true a description of the motion as another, although simplicity may be gained by choosing the base in a particular way.

But when motions are studied *causally*, it becomes evident that the choice of base is not merely a matter of convenience. It will, in fact, be seen that the "laws of motion" cannot be true independently of the base to which motion is referred.

518. Importance of Choice of Base in Applying the Laws of Motion. — That the laws of motion, if true for one base, cannot be true for another which moves in an arbitrary manner with respect to the first, may be shown by a simple illustration.

Conceive a horizontal platform to be rotating uniformly about a vertical axis fixed in the earth, and consider the motion of a ball which is placed upon the platform. Let this motion be specified both with respect to the platform and with respect to the earth, and let the two motions be compared.*

Suppose first that the ball is stationary with respect to the platform. If the platform be taken as base, the acceleration is zero; and if the laws of motion be applied, the inference is that the resultant of all forces acting upon the ball is zero, since the acceleration is always proportional to the resultant force. But if the earth be taken as base, the body describes a circle at a uniform speed. If ω is the

*A concrete idea of what is meant by the motion with respect to the platform may be obtained by considering how the motion will appear to an observer stationed upon the platform, whose view is so obstructed that the earth and the bodies fixed upon it are invisible to him. To his experience the platform is a "fixed" body, and is the only base available for estimating the motion of the ball. On the other hand, if a person stationed upon the earth can observe the ball, he may take the earth as his base for estimating its motion. The path described by the ball will appear quite different to the two observers, and its velocity and acceleration as estimated by them will have quite different values.

angular velocity of the platform and r the distance of the body from the axis of rotation, it has an acceleration with respect to the earth equal to $\omega^2 r$ directed toward the axis. The general equation

$$\text{force} = \text{mass} \times \text{acceleration}$$

therefore shows that the resultant of all forces acting upon the body is a force of magnitude $mr\omega^2$ directed toward the center of the circular path; m being the mass of the ball.

As another case, suppose the path of the ball upon the platform is a straight line passing through the axis of rotation, and that the velocity estimated with respect to the platform is constant. If the laws of motion hold when the platform is base, the resultant force acting upon the ball must be zero, since its acceleration is zero. But with respect to the earth the body describes a spiral with varying speed; hence if the laws of motion be applied with the earth as base, the inference is that the resultant of all forces acting upon the ball varies both in magnitude and in direction.

The actual forces acting upon the body are, however, the same whatever be the base of reference. Our conception of a force is that it is an action of one body upon another;* and the actions of other bodies upon the ball at any given instant cannot be supposed to depend upon whether the motion is estimated with respect to the earth or to the platform.

It is thus evident that if the laws of motion are true, and if our definition of force (which forms an essential element in the interpretation of those laws) is to be retained, it is not permissible to refer motions to a base chosen arbitrarily in applying the laws. It is necessary, in fact, to raise the fundamental question whether there is any base for which the laws of motion are true.†

519. Ultimate Base.— It is obvious that, unless there is a base for which the laws of motion are true, these laws are unintelligible.‡

---

*See Art. 212.

†It may be well to remark that this is not the same as the question whether such a base can be exactly determined by experiment.

‡Maxwell, in his discussion of Newton's first law ("Matter and Motion," Chapter III), argued that no alternative law can be stated which is intelligible "unless we admit the possibility of defining absolute rest and absolute velocity." And denying this possibility, he appeared to regard this as *a priori* evidence of the validity of the law. It would seem, however, that the argu-

That such a base is a reality must in fact be regarded as an essential part of the laws themselves.

Definition. — A body or frame of reference for which the laws of motion are true may be called an *ultimate base.*

520. Time. — In order to specify motion it is necessary to adopt some scale for estimating time. That is, some means must be adopted for comparing different intervals of time.

The first essential is to be able to specify successive *equal* intervals of time ; and for this purpose the only method practically available is to observe some body which is supposed to move uniformly, or to perform a definite series of motions in uniformly recurring cycles. Thus, we may take as equal the successive intervals occupied by complete revolutions of the earth upon its axis ; or the intervals occupied by successive vibrations of a pendulum.

Now the time-scale specified by the motion of one body will not in general agree exactly with that specified by the motion of another, and the question arises which, if any, of the scales thus defined is the "true" time-scale.

So far as a mere *description* of the motions of bodies is concerned, any time-scale will serve. The actual values of the velocity and acceleration of a particle will depend upon what scale is adopted, but a time-scale having been chosen, all motions can be described in terms of it. Some writers, in fact, assert that one time-scale gives as correct a description of motions as another, and that the only reason for preferring one to another is a gain of simplicity.

It is obvious, however, that the laws of motion cannot be true independently of the time-scale. Thus, the first law asserts that a body not acted upon by force (*i. e.*, not influenced by other bodies) would move uniformly in a straight line. But motion which is uniform when one time-scale is used may not be uniform when another is used.

The laws of motion are, in fact, wholly unintelligible unless we

---

ment which Maxwell applied to the negation of the law can be applied with equal validity to its affirmation. If the reasoning is valid, the conclusion appears to be that the laws of motion as stated by Newton are unintelligible.

That Newton's conception of absolute motion corresponds to a reality can hardly be denied, unless we are prepared to discard the whole Newtonian system. Whether the term "absolute" is properly applied is another question.

admit the reality of a time-scale for which they are true; and this reality must be accepted as a part of the laws themselves.*

521. **Absolute Space and Time.** — The reality of an ultimate base for specifying motion, and of an ultimate scale for estimating time, was explicitly affirmed by Newton. He in fact postulated *absolute space* and *absolute time* as the space and time implied in his statement of the laws of motion.

Many modern writers have found the Newtonian conception of absolute space and time so repugnant that they have discarded it. The attempt to destroy the force of Newton's arguments has not, however, been successful, and as already stated, there appears to be as much reason for accepting his conception of absolute motion as corresponding to reality as for accepting the Newtonian laws as a sound basis for the science of Mechanics.

522. **Absolute and Relative Motion.**—The motion of a body with respect to an ultimate base may be defined as its absolute motion; and its motion referred to any other than an ultimate base may be called its relative motion.†

In certain cases it is necessary to compare the motions of a particle as estimated with respect to different bases. In such cases it is convenient to call one of the motions absolute, even though the base of reference is not an ultimate one. Thus, there are many practical problems in which the motion of a particle with reference to the earth has to be compared with its motion referred to some body which is itself in motion with respect to the earth. The theory of turbine water-motors presents such a case, it being needful to estimate the motion of the particles of water both with respect to the rotating wheel and with respect to the earth. The latter may be called the absolute motion to distinguish it from the former. In such practical problems the earth is, in fact, the ultimate base to which we refer motions, although it is known not to be a true ultimate base as above defined. The error which results from regarding the earth as an ultimate base is for many purposes unimportant.‡

---

*It may be well to remark that to affirm the reality of a time-scale which is independent of the motions of actual bodies is not to affirm that we can practically attain to such a scale.

†However objectionable the word "absolute" may be in itself, its employment seems to be justified by usage.

‡A more definite discussion of the earth as a base is given in Art. 531.

§ 2. *Transformation from One Base to Another.*

523. **Change of Base.** — In the foregoing discussion the importance of the choice of a base for specifying the motion of a particle has been shown by simple illustrations, but the influence of the choice of base upon the values of the velocity and acceleration has not been explained in an exact manner. A mathematical discussion will now be given, showing how the values of the quantities which specify the motion are changed by changing from one base of reference to another having a given motion with respect to the first.

To discuss the problem in its full generality would be beyond the scope of this book, since the treatment already given of the motion of a particle and of a rigid body does not go beyond *plane motion* and *motion of translation.* The following discussion is therefore restricted in the same manner.

The case of translatory motion will be considered first; then a restricted case of plane motion (that in which the relative motion of the bases is a uniform rotation); and finally the general case of plane motion.

Of the two bases or frames to which the discussion refers, one may be called the primary, the other the secondary; and the motion referred to the former will often be called the absolute motion, while that referred to the latter is called the relative motion.

524. **Case in which the Relative Motion of the Bases is Translatory.** — Let the motion of a particle be referred to a base M', and also to a base M which itself has any motion of translation with respect to M'. Let x', y', z' be the coördinates of the particle referred to a set of rectangular axes $O'X'$, $O'Y'$, $O'Z'$, fixed in M'; x, y, z its coördinates with respect to axes OX, OY, OZ, fixed in M; and x_0, y_0, z_0 the coördinates of the origin O referred to the axes $O'X'$, $O'Y'$, $O'Z'$.

The position, velocity and acceleration of the particle with respect to M' are given by the values of x', y', z', and their first and second derivatives with respect to the time; the position, velocity and acceleration with respect to M by the values of x, y, z, and their first and second derivatives. Also, x_0, y_0, z_0, and their derivatives specify the position, velocity and acceleration of M relative to M'.

The relation between these different motions is expressed by the following equations:

$$\left.\begin{aligned} x' &= x_0 + x, \\ y' &= y_0 + y, \\ z' &= z_0 + z; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

$$\left.\begin{aligned} \dot{x}' &= \dot{x}_0 + \dot{x}, \\ \dot{y}' &= \dot{y}_0 + \dot{y}, \\ \dot{z}' &= \dot{z}_0 + \dot{z}; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

$$\left.\begin{aligned} \ddot{x}' &= \ddot{x}_0 + \ddot{x}, \\ \ddot{y}' &= \ddot{y}_0 + \ddot{y}, \\ \ddot{z}' &= \ddot{z}_0 + \ddot{z}; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

Equations (2) express the proposition that

The velocity of the particle with respect to the base M' is equal to the vector sum of its velocity with respect to M and the velocity of M with respect to M'.

Similarly, equations (3) state that

The acceleration of a particle with respect to the base M' is equal to the vector sum of its acceleration with respect to M and the acceleration of M with respect to M'.

Special case.—If M has a uniform rectilinear motion with respect to M', $\ddot{x}_0$, $\ddot{y}_0$, and $\ddot{z}_0$ are all zero. In this case *the acceleration of a particle with respect to M' is equal to its acceleration with respect to M.*

Examples.

1. A railway train moves uniformly on a straight track at the rate of 25 miles per hour. A passenger throws a stone so that to him it appears to move at right angles to the track with a velocity 20 ft.-per-sec. What is its velocity with respect to the earth?

2. In Ex. 1, how must the stone be thrown in order that its velocity referred to the earth shall be 20 ft.-per-sec. at right angles to the track?

3. A railway train moves along a straight track in such a way that in half a minute its speed increases uniformly from 10 miles per hour to 35 miles per hour. A stone dropped from a car window has what acceleration relative to the earth while falling to the ground? What acceleration has it relative to the train?

4. In Ex. 3, compute the velocity of the stone relative to the train after it has fallen 4 ft. vertically.

525. Case in which the Relative Motion of the Bases is a Uniform Rotation.*—Let the motion of a particle be referred to a rigid body or base M', and also to a body M which rotates uniformly about an axis fixed in M'. It will be convenient to speak of M' as a "fixed" body, the motion of the particle relative to it being called the "absolute" motion, and that relative to M the "relative" motion.

Let r denote the distance of the particle from the axis of rotation at time t; ω the angular velocity of the rotation of M relative to M'; v, p the relative velocity and acceleration; v', p' the absolute velocity and acceleration. It is required to find (*a*) the relation between v and v', and (*b*) the relation between p and p'. It will at first be assumed that the particle moves in a plane perpendicular to the axis of rotation.

(*a*) *Relation between velocities.*—Let Fig. 199 represent the plane of the motion, O being the point in which this plane is pierced by the axis of rotation. Let AB be the path traced by the particle on the rotating body during a short interval Δt. During that time this body turns through an angle $\omega\Delta t$, and AB is carried into a position $A'B'$ such that the angles AOA' and BOB' are each equal to $\omega\Delta t$. The path traced by the particle upon the fixed body M' is sòme curve AB'.

The velocity of the particle relative to the rotating body at the beginning of the interval Δt has the direction of the tangent to AB at A, and the magnitude

$$v = \lim\left[(\text{vector } AB)/\Delta t\right].$$

The absolute velocity at the same instant has the direction of the tangent to AB' at A, and the magnitude

$$v' = \lim\left[(\text{vector } AB')/\Delta t\right].$$

But $$\text{vector } AB' = \text{vector } AA' + \text{vector } A'B';$$

hence $$v' = \lim\left[(\text{vector } A'B')/\Delta t\right] + \lim\left[(\text{vector } AA')/\Delta t\right].$$

---

* This case is a special case of that treated in Art. 526. The geometrical discussion here given has the advantage of showing more vividly the relations of the vector quantities involved.

If u denotes the velocity of the point A of the body M due to the rotation,

$$\lim\left[(\text{vector } AA')/\Delta t\right] = u.$$

(The direction of u is perpendicular to AO and its magnitude is $r\omega$.) Also, in passing to the limit (Δt approaching 0), vector AB may replace vector $A'B'$. Hence

$$\text{vector } v' = \text{vector } v + \text{vector } u.$$

That is,

The absolute velocity of the particle is equal to the vector sum of its relative velocity and the velocity of that point of the rotating body which is the instantaneous position of the particle.

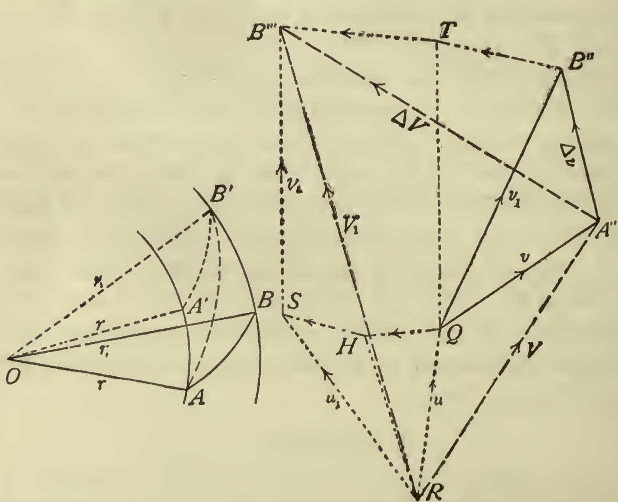


FIG. 199.

(b) *Relation between accelerations.*—In order to compare the absolute and relative accelerations, let the value of each be computed in the manner explained in Art. 250.

Let Δv denote the increment of the relative velocity v during the time Δt, and $\Delta v'$ the increment of v' during that time. Then

$$p = \lim\,(\Delta v/\Delta t)\;;$$

$$p' = \lim\,(\Delta v'/\Delta t).$$

We may first compare Δv and $\Delta v'$.

The value of Δv is found by drawing from some point, Q, vectors to represent the initial and final values of v. The former is parallel to the tangent to AB at A and is represented by QA'' (Fig. 199); the latter is parallel to the tangent to AB at B* and is represented by QB''; so that

$$\text{vector } A''B'' = \Delta v.$$

To find the value of $\Delta v'$ we must in like manner draw from some point vectors representing the initial and final values of v'. These are parallel to the tangents to the absolute path AB' at A and at B' respectively. Having drawn QA'' to represent the initial value of v, that of v' is found by combining with QA'' a vector RQ of magnitude $r\omega$ directed at right angles to OA (representing the velocity u of the point A of the rotating body). This gives RA'' as the vector value of the velocity v' at the beginning of Δt. Similarly, the final value is found as the sum of two components representing values of v and u at the end of Δt. We have already drawn QB'' to represent the final value of v; but since we are now referring velocities to the fixed body, the direction of this vector must be made parallel to the tangent to $A'B'$ at B'. That is, we take a vector QT equal in magnitude to QB'' but making with it an angle $\omega \Delta t$ (the angle turned through in time Δt by the tangent to AB at B). The other component of the final value of v' is perpendicular to OB' and of magnitude $OB' \cdot \omega$ or $r_1 \omega$. Represent this latter component by RS, and draw SB''' equal to vector QT; then RB''' represents the final value of v', and

$$\text{vector } A''B''' = \Delta v'.$$

It remains to compare Δv and $\Delta v'$.

The following vector equation may be written:

$$\Delta v' = \Delta v + B''T + TB''' = \Delta v + B''T + QS.$$

Also QS may be expressed as the sum of two component vectors QH, HS, the point H being located as follows: Make RH equal in length to RQ and angle $QRH = \omega \Delta t$; then RH is perpendicular to OA'. This construction makes the triangle RHS similar

* Although the path AB is at $A'B'$ at the end of the interval, QB'' is drawn parallel to the tangent at B in the original position, because when the rotating body is taken as base the relation between initial and final velocities is not affected by the rotation. The angle between these velocities is the angle between the tangents to AB at A and B.

to $OA'B'$, the corresponding sides being perpendicular each to each.

Replacing QS by $QH + HS$ in the last equation, dividing through by Δt and passing to the limit, Δt approaching 0,

$$p' = p + \lim[(B''T)/\Delta t] + \lim[(QH)/\Delta t] \\ + \lim[(HS)/\Delta t]. \qquad (2)$$

It is easy to evaluate the last three terms. Consider them in order.

(1) As Δt approaches 0, the angle $B''QT$ approaches 0 and $QB''T$ approaches a right angle, while QB'' approaches the direction QA''. Hence the limiting direction of the vector $B''T$ is perpendicular to QA'', that is, perpendicular to the relative velocity v at the time t. The limiting magnitude of $(B''T)/\Delta t$ may be found by considering the circular arc $B''T$ whose center is Q. Thus

$$\begin{aligned} \lim[(\text{chord } B''T)/\Delta t] &= \lim[(\text{arc } B''T)/\Delta t] \\ &= \lim[(v_1\omega\,\Delta t)/\Delta t] \\ &= \lim[v_1\omega] = v\omega. \end{aligned}$$

[v_1 is written for the value of v at the end of Δt.]

(2) In a precisely similar manner the limiting direction and magnitude of (vector QH)/Δt may be found from the triangle QRH. Its limiting direction is perpendicular to RQ, *i. e.*, it is AO. Its limiting magnitude is

$$RQ\cdot\omega = r\omega\cdot\omega = r\omega^2.$$

(3) The vector HS is perpendicular to the chord $A'B'$. As Δt approaches 0, $A'B'$ approaches AB, while chord AB approaches tangent to curve AB at A. Hence the limiting direction of HS is perpendicular to the relative velocity v. The limiting magnitude of $(HS)/\Delta t$ is found by comparing the similar triangles RHS, $OA'B'$. Each side of the former triangle is equal to ω times the corresponding side of the latter. That is,

$$\lim[(HS)/\Delta t] = \lim[(\omega\cdot A'B')/\Delta t] = \omega\lim[(A'B')/\Delta t] = \omega v.$$

Noticing that the first and third of the three vector quantities just evaluated are equal, it is seen that the last three terms of equation (2) are equivalent to the following two vector quantities:

An acceleration of magnitude $r\omega^2$ directed along the perpendicular drawn from the particle to the axis of rotation.

An acceleration $2\omega v$ directed at right angles to the relative velocity.

The vector equation (2), expressing the relation between p and p', may be written

$$p' = p + [r\omega^2] + [2v\omega], \quad . \quad . \quad . \quad (3)$$

the quantities in brackets being the magnitudes of the two vector quantities, and their directions being as just stated. It should be stated further that the direction of the component $[2v\omega]$ is that obtained by rotating the vector representing v through a right angle *in the direction in which the rotating body turns.*

Three-dimensional motion of a particle.—If the particle is not restricted to a plane perpendicular to the axis of rotation, Fig. 199 may still represent the projection of its motion upon such a plane. The propositions just deduced express the relation between the *resolved parts* of velocities and accelerations parallel to this plane. Thus, if α, α' denote the angles made by v, v' with the axis of rotation, and β, β' the angles made by p, p' with that axis, we may write instead of equations (1) and (3) the vector equations

$$[v' \sin \alpha'] = [v \sin \alpha] + u\,; \quad . \quad . \quad . \quad . \quad (1')$$

$$[p' \sin \beta'] = [p \sin \beta] + [r\omega^2] + [2\omega v \sin \alpha]. \quad (3')$$

Now the resolved part of v *parallel* to the axis is equal to that of v'; and by adding these equal vectors to the two members of (1′) we have

$$v' = v + u. \quad . \quad . \quad . \quad . \quad (1'')$$

Similarly, since the resolved parts of p and p' parallel to the axis are equal, we get from (3′)

$$p' = p + [r\omega^2] + [2\omega v \sin \alpha]. \quad . \quad (3'')$$

Equations (1″) and (3″) replace (1) and (3) when the motion of the particle is unrestricted. It is seen that (1″) is identical with (1), while (3″) differs from (3) only in the last term; this component of acceleration is always perpendicular to the axis of rotation, and is computed from the resolved part of v perpendicular to that axis.

EXAMPLES.

1. A horizontal platform rotates uniformly about a vertical axis fixed in the earth, at the rate of 100 rev.-per-min. A particle at rest on the platform 5 ft. from the axis of rotation has what velocity and what acceleration relative to the earth?

Ans. The magnitudes of the velocity and the acceleration are $v' = 52.4$ ft.-per-sec., $p' = 548$ ft.-per-sec.-per-sec.

2. A horizontal platform rotates with uniform angular velocity ω about a vertical axis fixed in the earth. A particle describes upon it a straight path passing through the axis, with uniform relative velocity v. Determine the velocity and the acceleration relative to the earth when the particle is distant r from the axis.

3. In Ex. 2 let $\omega = 20$ rev. per min., $v = 12$ ft. per sec. Solve for (*a*) $r = 0$, (*b*) $r = 5$ ft.

Ans. (*b*) $v' = 15.9$ ft. per sec., inclined $138° 54'$ to perpendicular from particle upon axis ; $p' = 54.8$ ft.-per-sec.-per-sec., at angle $66° 26'$ to same line.

4. In Ex. 2, let the shortest distance of the path from the axis be h, the other data being unchanged.

5. Assuming the platform to rotate as in Ex. 3, suppose a body to fall freely from rest under gravity from a point 8 ft. from the axis of rotation. Determine its velocity and acceleration relative to the platform after falling 0.5 sec.

526. Case in which the Relative Motion of the Bases is Any Plane Motion.—Let the motion of either base relative to the other be any plane motion, and let the particle be restricted to a plane common to the two bases, represented in Fig. 200. Let x, y be the coördinates of the particle referred to axes OX, OY in the secondary base M; x', y', its coördinates referred to axes $O'X'$, $O'Y'$ in the primary base M'; x_0, y_0 the coördinates of the origin O referred to $O'X'$, $O'Y'$; θ the angle between OX and $O'X'$.

The values of x', y' in terms of x, y, x_0, y_0 and θ are

$$x = x_0 + x \cos\theta - y \sin\theta. \qquad (1)$$

$$y' = y_0 + x \sin\theta + y \cos\theta. \qquad (2)$$

Differentiating with respect to the time,

$$\dot{x}' = \left[\dot{x}_0 - (x \sin\theta + y \cos\theta)\frac{d\theta}{dt}\right] + \left[\dot{x} \cos\theta - \dot{y} \sin\theta\right], \quad (3)$$

$$\dot{y}' = \left[\dot{y}_0 + (x \cos\theta - y \sin\theta)\frac{d\theta}{dt}\right] + \left[\dot{x} \sin\theta + \dot{y} \cos\theta\right]. \quad (4)$$

A second differentiation gives

$$\ddot{x}' = \left[\ddot{x}_0 - (x \cos\theta - y \sin\theta)\left(\frac{d\theta}{dt}\right)^2 - (x \sin\theta + y \cos\theta)\frac{d^2\theta}{dt^2}\right]$$
$$- 2\left[(\dot{x} \sin\theta + \dot{y} \cos\theta)\frac{d\theta}{dt}\right] + \left[\ddot{x} \cos\theta - \ddot{y} \sin\theta\right], \quad (5)$$

$$\ddot{y}' = \left[\ddot{y}_0 - (x\sin\theta + y\cos\theta)\left(\frac{d\theta}{dt}\right)^2 + (x\cos\theta - y\sin\theta)\frac{d^2\theta}{dt^2}\right]$$
$$+ 2\left[(\dot{x}\cos\theta - \dot{y}\sin\theta)\frac{d\theta}{dt}\right] + \left[\ddot{x}\sin\theta + \ddot{y}\cos\theta\right] \quad (6)$$

Let v and p represent the velocity and acceleration of the particle referred to the base M, and v', p' the velocity and acceleration referred to the base M'.

Then v is equivalent to components $\dot{x}$, $\dot{y}$ in directions OX, OY; and p to components $\ddot{x}$, $\ddot{y}$ in these directions.

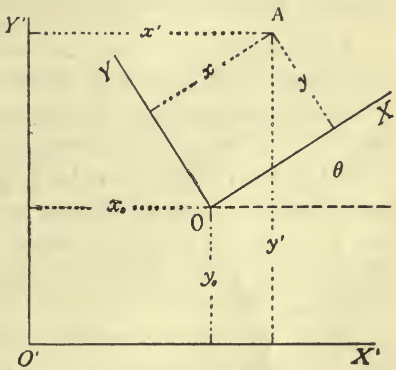


FIG. 200.

Also v' is equivalent to components $\dot{x}'$, $\dot{y}'$ in directions $O'X'$, $O'Y'$; and p' to components $\ddot{x}'$, $\ddot{y}'$ in these directions.

If v and v' were equal in magnitude and direction, the value of $\dot{x}'$ would equal the sum of the components of $\dot{x}$ and $\dot{y}$ in the direction $O'X'$, being therefore given by the last two terms of equation (3); while the value of $\dot{y}'$ would be given by the last two terms of equation (4). The remaining terms in the values of $\dot{x}'$ and $\dot{y}'$ as given by these equations therefore represent the difference between v and v'; that is, they represent the components in directions $O'X'$ and $O'Y'$ of the vector which, added to v, gives v'.

Similiarly, the component of p in the direction $O'X'$ is given by the last two terms in equation (5), while its component in the direction $O'Y'$ is given by the last two terms of (6). The remaining terms in the second members of these equations, therefore, represent the difference between p and p'; that is, they represent the components in the directions $O'X'$ and $O'Y'$ of the vector which added to p gives p'.

It remains to interpret these results.

Interpretation of result in case of velocities.—The first three terms in the second member of equation (3) give the values of $\dot{x}'$ for a particle fixed to the base M, as is seen by the fact that they are

the terms which remain when x and y are constant. Similarly for the first three terms in the value of $\dot{y}'$. Hence it is seen that the vector which represents the difference between v and v' is equal to *the velocity with respect to the base M' of that point of M which momentarily coincides with the particle.* We have therefore the general result that

The velocity of a particle referred to any base M' is equal to the vector sum of its velocity referred to another base M and the velocity (referred to M') of that point of M which momentarily coincides with the particle.

Interpretation of result in case of acceleration.—The result is less simple in the case of accelerations.

Notice first that the acceleration, with respect to the primary base M', of that point of the secondary base M which is momentarily the position of the particle, is given by the terms remaining in the values of $\ddot{x}'$ and $\ddot{y}'$ when x and y are constant; that is, the first five terms in the second member of (5) represent its component in the direction $O'X'$, and the corresponding terms in (6) its component in the direction $O'Y'$.

Again, the last two terms in (5), and the corresponding terms in (6), represent the resolved parts of p in the directions $O'X'$, $O'Y'$ respectively.

There remain to be interpreted two terms in each of the two equations. In other words, we have to consider what vector it is whose resolved parts are

$$\left.\begin{array}{ll} -2\dfrac{d\theta}{dt}(\dot{x}\sin\theta + \dot{y}\cos\theta) & \text{in direction } O'X', \\ +2\dfrac{d\theta}{dt}(\dot{x}\cos\theta - \dot{y}\sin\theta) & \text{`` \quad ``} \quad O'Y'. \end{array}\right\} \qquad (7)$$

Now $\qquad \dot{x}\cos\theta - \dot{y}\sin\theta =$ resolved part of v in direction $O'X'$,

$\qquad \dot{x}\sin\theta + \dot{y}\cos\theta =$ " " " v " " $O'Y'$;

hence the quantities (7) are the components of a vector perpendicular to v, of magnitude $2v(d\theta/dt) = 2v\omega$, if ω denotes the angular velocity of the base M with respect to M'.

It is thus seen that equations (5) and (6) are equivalent to the following proposition:

The acceleration of a particle relative to any base M' is equal to the vector sum of three components: (1) its acceleration relative to another base M having any plane motion with respect to M', (2) the acceleration relative to M' of that point of M which momentarily coincides with the particle, and (3) an acceleration directed at right angles to the velocity of the particle relative to M, of magnitude equal to twice the product of that velocity into the angular velocity of M with respect to M'. The direction of this third component is that assumed by the vector representing the velocity relative to M if turned through 90° in the direction in which M rotates relatively to M'.

The propositions just proved may be concisely expressed by vector equations. Thus, if u denotes the velocity relative to M' of that point of M which instantaneously coincides with the particle, the relation between velocities is

$$v' = v + u. \quad . \quad . \quad . \quad . \quad (1''')$$

And if the acceleration components numbered (2) and (3) in the above statement be represented by vector symbols p'' and p''' respectively, the relation between accelerations is

$$p' = p + p'' + p'''. \quad . \quad . \quad . \quad (3''')$$

These are seen to include equations $(1'')$ and $(3'')$ of Art. 525.

Examples.

1. Show that the above general relation between accelerations reduces to that given in Art. 524 when the relative motion of the two bases is a translation with uniform velocity.

2. Apply the general result to the case in which the relative motion of the two bases is a uniform rotation about an axis fixed in both, and compare with the result reached in Art. 525.

3. In what cases does the component p''' vanish?

4. A wheel of radius a rolls uniformly along the ground, the velocity of its center being V. Determine the velocity and acceleration relative to the earth of a point in the circumference (a) when in its highest position, (b) when in its lowest position.

5. In Ex. 4, suppose a particle to move uniformly along a diameter of the wheel, and that its relative path is vertical when it passes through the center. Determine its velocity and acceleration relative to the earth at that instant.

527. Three-Dimensional Motion of a Particle.—The foregoing discussion assumes the particle to move in a plane parallel to the relative motion of the bases. In order to remove this restriction, let the point A (Fig. 200) represent the orthographic projection of the position of the particle upon a plane parallel to the relative motion of the bases. Reasoning as in Art. 525, it is seen that

When the motion of the particle is unrestricted, the relation between velocities is the same as in the case of plane motion; while the relation between accelerations is changed by the substitution for the velocity relative to M of the resolved part of that velocity parallel to the plane of the relative motion of the bases. This change affects only the component p'''.

EXAMPLE.

Show that the above proposition holds when the relative motion of the bases is equivalent to a plane motion combined with any translatory motion.

§ 3. *Dynamics of Relative Motion.*

528. Equation of Motion for Any Base.—The importance of the choice of a base of reference for the purpose of applying the laws of motion was shown in Art. 518. This is made more definite by the foregoing analysis. If the equation

$$\text{force} = \text{mass} \times \text{acceleration}$$

holds when the motion of a particle is referred to a base M', it will not hold when the motion is referred to another base M, unless the acceleration has the same value for the two bases; and this will be true *only if the motion of M relative to M' is a translation with uniform rectilinear velocity*.

It is, however, easy to write a dynamical equation which will hold for any base which itself moves in a known manner with respect to an ultimate base. Thus, applying the result of Art. 526, let the base M' be an ultimate base, and let P' denote the resultant of all forces acting upon the particle, the remaining notation being as before. The general equation of motion is

$$P' = mp' ; \quad . \quad . \quad . \quad . \quad (1)$$

or, by equation (3''') of Art. 526, we may write the vector equation

$$P' = mp + mp'' + mp'''. \quad . \quad . \quad . \quad (2)$$

This equation serves to determine p when the forces are known, if the motion of M relative to M' is known so that p'' and p''' can be computed.

529. Fictitious Forces.—The last equation may be written

$$P' - mp'' - mp''' = mp. \qquad (3)$$

Or, if P is written for the vector $P' - mp'' - mp'''$,

$$P = mp. \qquad (4)$$

The equation of motion referred to the secondary base M thus has the same form as that referred to an ultimate base, P taking the place of the "resultant force" acting upon the particle. But the forces of which P is the resultant include, besides the actual forces (whose resultant has been called P'), two "fictitious" forces whose values ($-mp''$ and $-mp'''$) depend upon the motion of the secondary base with respect to an ultimate base.

Equations of equilibrium.—If the particle is at rest relative to the base M, the acceleration p''' is 0, and the only fictitious force to be introduced into the equation of equilibrium is the force $-mp''$.

EXAMPLES.

[In these examples it may be assumed that the earth is an ultimate base. The error involved in this assumption will be considered in Art. 531.]

1. A ball is thrown back and forth by two persons on the deck of a steamer which moves with uniform speed in a constant direction. How does the motion of the boat affect the game? What is the effect of a change in the speed of the steamer?

2. An elevator platform starts from rest with an upward acceleration $g/4$, then moves upward with constant velocity, then has downward acceleration $g/6$. Taking the platform as base, write the equation of motion for a body resting upon the platform, for each of the three periods of the motion.

3. An elevator platform, starting from rest, moves with a downward acceleration which increases uniformly from 0 to $2g$ in a period of 4 seconds. (a) Write the equation of motion for a body resting on the platform, taking the platform as base. (b) Determine completely the motion of the body during 4 seconds. (c) How far will it be from the platform at the end of that time?

4. A horizontal platform rotates uniformly about a fixed vertical axis. Write the equations of motion for a body at rest upon the platform, taking the latter as base.

Ans. The actual forces acting upon the body are (1) its weight vertically downward, and (2) the supporting force exerted by the platform. Of the fictitious forces, one vanishes, since the velocity relative to the platform is zero; the other, $-mp''$, is a force directed at right angles to the axis of rotation and away from it. If r is the distance of the particle from the axis of rotation and ω the angular velocity of the platform, $p'' = \omega^2 r$.

5. In Ex. 2. of Art. 525, assuming the earth to be an ultimate base, (*a*) determine the resultant of the actual forces acting upon the particle when distant r from the axis. (*b*) What fictitious forces must be assumed to act if the platform is taken as base?

6. A particle slides in a smooth straight tube which rotates uniformly about a vertical axis at right angles to its length. The actual forces acting upon the particle being its weight and the pressure of the tube, (*a*) write the equation of motion, taking the tube as base. (*b*) If the particle passes through the axis of rotation with velocity V relative to the tube, what will be its velocity when distant r from this position?

Ans. (*b*) $v^2 = V^2 + r^2\omega^2$, ω being the angular velocity.

7. Solve Ex. 6 assuming the tube to be inclined at angle α to the horizontal, the axis still being vertical.

Ans. (*b*) $v^2 = V^2 + r^2\omega^2 \cos^2 \alpha + 2gr \sin \alpha$.

530. Centrifugal Force.—When the secondary base rotates uniformly about a fixed axis, the fictitious force $-mp''$ is directed away from the axis of rotation, and its magnitude is $m\omega^2 r$, ω being the angular velocity and r the distance of the particle from the axis. This is called the *centrifugal force* due to the rotation of the base to which the motion is referred.*

Whatever motion the secondary base may have with respect to an ultimate base, the component of the fictitious force $-mp''$ directed away from the instantaneous axis may be called the centrifugal force.

531. The Earth as Base.—The earth is usually the most convenient base for estimating the motions of terrestrial bodies. The fictitious forces which must be introduced into the equations of motion in order to take account of the known motions of the earth are for many purposes negligible in comparison with the actual forces.

---

* This is the proper use of the term centrifugal force, to which allusion was made in the foot-note on p. 271.

The most important of these motions of the earth is the diurnal rotation.

Correction for earth's rotation.—Let a frame of reference be taken consisting of three rectangular axes intersecting at the earth's center, one coinciding with the axis of rotation, a second lying in the plane through that axis and a fixed star, the third perpendicular to this plane. The motion of the earth with respect to this frame is a rotation with uniform angular velocity $\omega = 0.00007292$ rad. per sec. (Ex. 1, p. 275). The effect of this rotation upon the motion of a particle relative to the earth is to give it two components of acceleration which are the negatives of p'' and p''' as defined in Art. 526. In other words, the fictitious forces to be introduced are $-mp''$ and $-mp'''$. The former of these is the "centrifugal" force whose magnitude is $m\omega^2 r$, directed away from the axis of rotation; the latter is a force of magnitude $2m\omega v \sin \alpha$, α being the angle between v and the axis of rotation. This latter force is always perpendicular to the axis of rotation and to the direction of the relative motion, in such a way that the particle is deflected *toward the right* in the northern hemisphere and *toward the left* in the southern, as seen by a person standing upon the earth and facing in the direction of the relative motion of the particle.

The greatest value of the acceleration p'' occurs at the equator, being 3.391 C. G. S. units* or 1/289 of g. Practically, the acceleration $-p''$ cannot be separated from the acceleration due to gravity except by computation. The value of g as determined experimentally at any place is the resultant of the component $-p''$ and the acceleration due to the attraction of the earth. In practice, therefore, the centrifugal force due to the earth's rotation is always taken account of when the weight of the body is one of the forces entering the equation of motion.

The acceleration p''' is small unless the velocity of the particle relative to the earth is very great. If $v \sin \alpha = 100$ meters per second, $p''' = 1.458$ C. G. S. units.

The deflection of a particle toward the right in the northern hemisphere and toward the left in the southern has an important effect upon the circulation of the atmosphere. It is this which determines the direction of rotation of cyclonic disturbances.

---

* See Ex. 3, p. 275.

Correction for motions of the earth about the sun and the moon.—The origin of the frame of reference described at the beginning of this Article is at the center of the earth, and the frame has therefore a motion of translation due to the orbital motion of the earth's center of mass. By reason of the attraction of the sun, the center of mass of the earth describes an approximately circular orbit about the common center of mass of sun and earth; and by reason of the moon's attraction it describes an approximately circular orbit about the common center of mass of moon and earth. These are most simply considered separately. The fictitious forces which must be assumed to act upon any terrestrial particle because of these motions, if the frame of reference above described be taken as base, are easily computed from the general principle of Art. 524. The most important application in which these corrections need to be made is in the theory of the tides.

INDEX.

Hoskins